SOME REMARKS ON UNCOUNTABLE RAINBOW RAMSEY THEORY

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Abstract. We discuss the rainbow Ramsey theorems at limit cardinals and successors of singular cardinals, addressing some questions in [2] and [1]. In particular, we show for inaccessible κ, $\kappa \to^{poly} (\kappa)^2_{2-bdd}$ does not characterize weak compactness and for singular κ, GCH$+\square_\kappa$ implies $\kappa^+ \not\to^{poly} (\eta)^2_{<\kappa-bdd}$ for any $\eta \geq cf(\kappa)^+$ and $\kappa^+ \to^{poly} (\nu)^2_{<\kappa-bdd}$ for any $\nu < cf(\kappa)^+$. We also provide a simplified construction of a model for $\omega_2 \not\to^{poly} (\omega_1)^2_{2-bdd}$ originally constructed in [1] and show the witnessing coloring is indestructible under strongly proper forcings but destructible under some c.c.c forcing. Finally, we conclude with some remarks and questions on possible generalizations to rainbow partition relations for triples.

Contents

1. Introduction

Fix ordinals λ, i, κ and $n \in \omega$.

Definition 1.1. We use $\lambda \to (\kappa)^n_i$ to abbreviate: for any $f : [\lambda]^n \to i$, there exists $A \subset \lambda$ of order type κ such that $f \restriction [A]^n$ is a constant function. Such A is called a *monochromatic* subset of λ (with respect to f).

Definition 1.2. We use $\lambda \to^{poly} (\kappa)^n_{i-bdd}$ to abbreviate: for any $f : [\lambda]^n \to \lambda$ that is *i-bounded*, namely for any $\alpha \in \lambda$, $|f^{-1}\{\alpha\}| \leq i$, there exists $A \subset \lambda$ of order type κ such that $f \restriction [A]^n$ is injective. Such A is called a *rainbow* subset of λ (with respect to f).

Remark 1.3. $\to^{poly}$ is sometimes denoted as $\to^*$. We adopt $\to^{poly}$ to avoid possible confusion, as rainbow subsets are sometimes called "polychromatic" subsets.

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$\lambda \to (\kappa)^n_i$ implies $\lambda \to^{poly} (\kappa)^n_{i-bdd}$ as given a i-bounded coloring it is possible to cook up a dual i-coloring for which any monochromatic subset will be a rainbow subset for the original coloring. This is the *Galvin's trick*. This explains why rainbow Ramsey theory is also called sub-Ramsey theory in finite combinatorics.

In many cases, the rainbow analogue is a strict weakening. For example:

1 In finite combinatorics, the sub-Ramsey number $sr(K_n, k)$, which is the least m such that $m \to^{poly} (n)^2_{k-bdd}$, is bounded by a polynomial in n and k (Alspach, Gerson, Hahn and Hell [3]). This is in contrast with the Ramsey number which grows exponentially.

2 In reverse mathematics, over RCA_0, $\omega \to^{poly} (\omega)^2_{2-bdd}$ does not imply $\omega \to (\omega)^2_2$ (Csima and Mileti [5]).

3 In combinatorics on countably infinite structures, the Rado graph is Rainbow Ramsey but not Ramsey (Dobrinen, Laflamme, and Sauer [6]).

4 In combinatorics on the ultrafilters on ω, Martin's Axiom implies there exists a Rainbow Ramsey ultrafilter that is not a Ramsey ultrafilter (Palumbo [12]).

5 In uncountable combinatorics, ZFC proves $\omega_1 \not\to (\omega_1)^2_2$ but $\omega_1 \to^{poly} (\omega_1)^2_{2-bdd}$ is consistent with ZFC (Todorcevic [14]).

Results in this note serve as further evidence that rainbow Ramsey theory is a strict weakening of Ramsey theory. We focus on the area of uncountable combinatorics.

The organization of the paper is:

(1) In Section 2, we discuss rainbow Ramsey theorems at limit cardinals. In particular, we show $\kappa \to^{poly} (\kappa)^2_{2-bdd}$ for an inaccessible cardinal κ does not imply κ is weakly compact, answering a question in [2];

(2) In Section 3, we discuss the rainbow Ramsey theorems at the successor of singular cardinals. Answering a question in [1], we show $\mathrm{GCH} + \square_\kappa$ implies $\kappa^+ \not\to^{poly} (\eta)^2_{<\kappa-bdd}$ for any $\eta \geq cf(\kappa)^+$ and $\kappa^+ \to^{poly} (\nu)^2_{<\kappa-bdd}$ for any $\nu < cf(\kappa)^+$.

(3) In Section 4, we use the method of Neeman developed in [11] to simplify the construction of a model by Abraham and Cummings [1] in which $\omega_2 \not\to^{poly} (\omega_1)^2_{2-bdd}$. Furthermore, we show in this model, the witnessing coloring is indestructible under strongly proper forcings but destructible under c.c.c forcings. In other words, the coloring witnessing $\omega_2 \not\to^{poly} (\omega_1)^2_{2-bdd}$ remains the witness to the same negative partition relation in any strongly proper forcing extension but there exists a c.c.c forcing extension that adds a rainbow subset of size ω_1 for that coloring. As a result, $\omega_2 \not\to^{poly} (\omega_1)^2_{2-bdd}$ is compatible with the continuum being arbitrarily large.

(4) In Section 5, we briefly discuss possibilities and restrictions of generalizations to partition relations for triples.

2. Rainbow Ramsey at limit cardinals

In [2], Abraham, Cummings and Smyth studied the rainbow Ramsey theory at small uncountable cardinals and successors of regular cardinals. They asked what can be said about the rainbow Ramsey theory at inaccessible cardinals. A test question they asked was for any inaccessible cardinal κ, whether $\kappa \to^{poly} (\kappa)^2_2$

characterize weak compactness. We answer this in the negative. Fix a regular uncountable cardinal κ.

Definition 2.1. We say $f : [\kappa]^n \to \kappa$ is a normal coloring if whenever $\bar{a}, \bar{b} \in [\kappa]^n$ are such that $f(\bar{a}) = f(\bar{b})$, then $\max \bar{a} = \max \bar{b}$.

Definition 2.2. A normal function $f : [\kappa]^2 \to \kappa$ is regressively bounded (reg-bdd) if there exists $\lambda < \kappa$ such that $\kappa \cap cof(\geq \lambda)$ is stationary in κ and for all $\alpha \in \kappa \cap cof(\geq \lambda)$, and $i < \kappa$, $\{\beta \in \alpha : f(\beta, \alpha) = i\}$ is bounded in α. We use $\kappa \to^{poly} (\kappa)^2_{reg-bdd}$ to denote the statement: for any normal regressively bounded $f : [\kappa]^2 \to \kappa$, there exists a subset $A \in [\kappa]^\kappa$ such that A is a rainbow subset for f.

Remark 2.3. Notice for any weakly inaccessible cardinal κ and any cardinal $\lambda < \kappa$, $\kappa \to^{poly} (\kappa)^2_{reg-bdd}$ implies $\kappa \to^{poly} (\kappa)^2_{\lambda-bdd}$. To see this, given $f : [\kappa]^2 \to \kappa$ that is λ-bounded, recursively, we may find a subset $B \in [\kappa]^\kappa$ such that $f \restriction [B]^2$ is normal. Hence without loss of generality we may assume f is normal. Then it is easy to see that f is regressively bounded witnessed by λ^+.

Remark 2.4. Even though we cannot employ Galvin's trick of dual colorings since there may not be any $\lambda < \kappa$ that bounds the sizes of color classes, we do have that if κ is weakly compact, then $\kappa \to^{poly} (\kappa)^2_{reg-bdd}$.

It turns out that weak compactness is not necessary. More precisely, the existence of certain ideal on κ will ensure $\kappa \to^{poly} (\kappa)^2_{reg-bdd}$. In some sense, κ is a "generic large cardinal" (for more on this topic, see [8]).

Definition 2.5. Let κ, λ, η be cardinals. $I \subset P(\kappa)$ an ideal on κ is

- non-trivial if $\kappa \notin I$;
- λ-complete if for any $\alpha < \lambda$ and $\{X_i \in I : i < \alpha\}$, $\bigcup_{i<\alpha} X_i \in I$;
- η-saturated if $P(\kappa)/I$ has η-c.c, in other words, for any collection $X \subset I^+$ with $|X| \geq \eta$, there exist $A, B \in X$ such that $A \cap B \in I^+$;
- normal if for any $A \in I^+ =_{def} P(\kappa) - I$ and any regressive function f defined on A, there exists $B \subset A$ and $B \in I^+$ such that $f \restriction B$ is a constant function.

We list some standard facts, which can be found in [8].

Fact 2.6. *Fix a κ-saturated κ-complete normal ideal I on κ. Let G be a generic ultrafilter on $\mathbb{P} = P(\kappa)/I$ over V then*

(1) I is precipitous, namely, in $V[G]$, the ultrapower $Ult(V, G) = \{[f]_G : f \in V\}$ is well-founded. Let $j : V \to M \simeq Ult(V, U)$ be the ultrapower map in $V[G]$ and M is the transitive collapse of $Ult(V, U)$.
(2) $V[G] \models {}^\kappa M \subset M$.
(3) G is V–κ-complete, meaning for any $\alpha < \kappa$ and any $\langle A_i : i < \alpha\rangle \in V$ with $A_i \in G$ for all $i < \alpha$, $\bigcap_{i<\alpha} A_i \in G$.
(4) G is V-normal, namely for any $A \in V$, $A \in G$ iff $\kappa \in j(A)$.

Theorem 2.7. *If a regular cardinal κ carries a non-trivial normal κ-saturated κ-complete ideal, then $\kappa \to^{poly} (\kappa)^2_{reg-bdd}$.*

Proof. Fix a regressively bounded normal coloring $f : [\kappa]^2 \to \kappa$ witnessed by $\lambda < \kappa$ and a κ-saturated κ-complete normal ideal I on κ. Call $F \in [\kappa]^{<\kappa}$ *promising* if $A(F) =_{def} \{\alpha < \kappa : F \cup \{\alpha\} \text{ is rainbow}\} \in I^+$. Note that if F is promising then F is rainbow.

Claim 2.8. *If F is promising, then there exists $\bar{\gamma} \in A(F), \bar{\gamma} > F$ such that*

$$A(F) - A(F \cup \{\bar{\gamma}\}) \in I.$$

In particular, $F \cup \{\bar{\gamma}\}$ is promising.

Proof of the claim. Let $G \subset P(\kappa)/I$ be the generic ultrafilter over V. In $V[G]$, let $j : V \to M \simeq Ult(V, G)$ be the generic ultrapower embedding. In M, by elementarity $j(f)$ is regressively bounded as witnessed by $\lambda < \kappa$. Since $\kappa > \lambda$ is regular in M, for each $\alpha \in F$, there exists $\gamma_\alpha \in \kappa$ such that any $\eta > \gamma_\alpha$, $j(f)(\alpha, \kappa) \neq j(f)(\eta, \kappa)$. Let $\gamma = \sup_{\alpha \in F} \gamma_\alpha < \kappa$. Pick a name $\dot{\gamma}$ for this ordinal such that $\Vdash_{P(\kappa)/I} \forall \eta > \dot{\gamma}$ and $\alpha \in F$, $j(f)(\alpha, \kappa) \neq j(f)(\eta, \kappa)$. Since $P(\kappa)/I$ is κ-c.c and $A(F)$ is unbounded in κ, there exists $\bar{\gamma} \in A(F)$ and $\bar{\gamma} > \max F$ such that $\Vdash_{P(\kappa)/I} \dot{\gamma} < \bar{\gamma}$.

We claim $\bar{\gamma}$ is as desired. Suppose for the sake of contradiction that $A(F) - A(F \cup \{\bar{\gamma}\}) \in I^+$. Let $H \subset P(\kappa)/I$ be generic containing $A(F) - A(F \cup \{\bar{\gamma}\})$ with the associated generic elementary embedding j. By normality, we know $\kappa \in j(A(F) - A(F \cup \{\bar{\gamma}\}))$. This means $j(f) \restriction [F \cup \{\kappa\}]^2$ and $j(f) \restriction [F \cup \{\bar{\gamma}\}]^2$ are injective but $j(f) \restriction [F \cup \{\bar{\gamma}, \kappa\}]^2$ is not injective. Therefore, there exist $\alpha \in F$ such that $j(f)(\bar{\gamma}, \kappa) = j(f)(\alpha, \kappa)$. But this contradicts with the choice of $\bar{\gamma}$.

$\square$

Recursively we will construct $\langle F_i : i < \kappa \rangle$ such that

(1) $F_0 = \emptyset$;
(2) for each $i < j < \kappa$, $F_i \subsetneq F_j$;
(3) for each limit $\delta < \kappa$, $F_\delta = \bigcup_{j<\delta} F_j$;
(4) for each $i < \kappa$, F_i is promising;
(5) for each $i < \kappa$, $A(F_i) - A(F_{i+1}) \in I$.

At the successor stage $\beta + 1$, apply Claim 2.8 to get $F_{\beta+1} \supsetneq F_\beta$ such that $A(F_\beta) - A(F_{\beta+1}) \in I$. At the limit stage $\delta < \kappa$, let $F_\delta = \bigcup_{j<\delta} F_j$. We need to verify that F_δ is promising. We claim that

$$A(F_\delta) \supset \kappa - (\bigcup_{j<\delta} (A(F_j) - A(F_{j+1}))) \backslash \sup F_\delta + 1.$$

To see this, fix any $\lambda \in \kappa - (\bigcup_{j<\delta} (A(F_j) - A(F_{j+1}))), \lambda > F_\delta$ and suppose for the sake of contradiction that $\lambda \notin A(F_\delta)$. There exist $a, b \in F_\delta$ such that $f(a, \lambda) = f(b, \lambda)$. Let $j < \delta$ be the least such that there exist $a, b \in F_j$ with $f(a, \lambda) = f(b, \lambda)$. Also note that j must be a successor ordinal, say $j = k + 1$. $\lambda \notin A(F_{k+1})$ but $\lambda \in A(F_k)$ by minimality of j. Hence $\lambda \in A(F_k) - A(F_{k+1})$, contradicting with the assumption about λ. By the κ-completeness of I, $A(F_\delta) \in I^+$ so F_δ is promising.

Finally let $F = \bigcup_{j<\kappa} F_j$, which is a desired rainbow subset for f of size κ. $\square$

Remark 2.9. Kunen in [10] showed that it is consistent relative to a measurable cardinal that there exists an inaccessible cardinal κ that carries a non-trivial κ-complete κ-saturated normal ideal but κ is not weakly compact.

However, for our purpose, we can get the $\kappa \to^{poly} (\kappa)^2_{reg-bdd}$ from the existence of a weakly compact cardinal. The reason is that in the proof of Theorem 2.7, it is sufficient when the domain of the generic embedding is a large enough fragment of V instead of V itself. More precisely, what we need is that for any κ-model N, namely ${}^{<\kappa}N \subset N$, $\kappa \in N, |N| = \kappa$ and N is the transitive collapse of some $X \prec H(\theta)$ for some sufficiently large regular θ, there exists a κ-c.c. forcing $\mathbb{P}$ such that $\Vdash_{\mathbb{P}}$ there exists a transitive κ-model M and an elementary embedding $j : N \to M$ with critical point κ. We can arrange this by first making the weakly compact cardinal indestructible under $Add(\kappa, 1)$ (or equivalently indestructible under $Add(\kappa, \lambda)$ for any λ), and then use the theorem of Kunen that $Add(\kappa, 1)$ is forcing equivalent to $P * \dot{T}$ where P adds a homogeneous κ-Suslin tree $\dot{T}$.

Remark 2.10. The Kunen model also shows that the existence of a κ-Suslin tree is consistent with $\kappa \to^{poly} (\kappa)^2_{reg-bdd}$. The existence of a κ-Suslin tree is sometimes strong enough to refute some weak consequences of $\kappa \to (\kappa)^2_2$. For example Todorcevic proved in [13] that for any regular uncountable cardinal κ, the existence of κ-Suslin tree implies $\kappa \not\to [\kappa]^2_\kappa$, namely there exists a coloring $f : [\kappa]^2 \to \kappa$ such that any $X \in [\kappa]^\kappa$, $f''[X]^2 = \kappa$.

Corollary 2.11. *It is consistent relative to a weakly compact cardinal that for some inaccessible cardinal κ that is not weakly compact, $\kappa \to^{poly} (\kappa)^2_{\lambda-bdd}$ for any $\lambda < \kappa$.*

Corollary 2.12. *If κ is real-valued measurable, then $\kappa \to^{poly} (\kappa)^2_{\lambda-bdd}$ for any $\lambda < \kappa$.*

Corollary 2.13. *If κ is weakly compact, then $\kappa \to^{poly} (\kappa)^2_{reg-bdd}$ is indestructible under any forcing satisfying λ-c.c. for some $\lambda < \kappa$.*

The trick of using some large enough ordinal to "guide" the construction can also be used analogously to prove the following, which provides more contrast with its dual Ramsey statement:

Lemma 2.14. *For any singular strong limit κ, $\kappa \to^{poly} (\kappa)^2_{\lambda-bdd}$ for any $\lambda < \kappa$.*

Remark 2.15. Given a λ-bounded coloring f on $[\kappa]^2$, we claim that there is $B \in [\kappa]^\kappa$ such that $f \restriction [B]^2$ is normal. Fix a continuous sequence of strictly increasing regular cardinals $\langle \kappa_i : i < cf(\kappa)\rangle$ with $\kappa_0 > \max\{cf(\kappa), \lambda\}$ converging to κ. We find $\langle A_i : i < cf(\kappa)\rangle$ such that

- for any $i < cf(\kappa)$, $A_i \subset \kappa_i$ and $|A_i| = \kappa_i$
- for any $i < j < cf(\kappa)$, $A_i \subsetneq A_j$
- for any limit $\delta < cf(\kappa)$, $A_\delta = \bigcup_{i<\delta} A_i$
- for any $i < cf(\kappa)$, $f \restriction [A_i]^2$ is normal

The construction clearly gives $B = \bigcup_{i<cf(\kappa)} A_i$ such that $f \restriction [B]^2$ is normal. The construction at limit stages is clear. At stage $i + 1$, we inductively find a subset $C \subset \kappa_{i+1} - \kappa_i$ of size κ_{i+1} such that $f \restriction [A_i \cup C]^2$ is normal. Suppose we have built $C' \subset \kappa_{i+1} - \kappa_i$ of size $\leq \kappa_i$, we demonstrate how to add one more element. As $|C' \cup A_i| \leq \kappa_i, \lambda < \kappa_i$ and κ_{i+1} is regular, there exists $\gamma > \max C' + 1$ such that there do not exist $a \in [C' \cup A_i]^2, \beta \in C' \cup A_i$ with $f(a) = f(\beta, \gamma)$. It is easy to see that $f \restriction [A_i \cup C' \cup \{\gamma\}]^2$ is normal.

Proof of Lemma 2.14. Fix a λ-bounded coloring $f : [\kappa]^2 \to \kappa$. By the remark above, we may assume f is normal. Let $\eta = cf(\kappa)$. Fix an increasing sequence of regular cardinals $\langle \kappa_i : i < \eta\rangle$ such that

(1) $\kappa_0 > \max\{\lambda, \eta\}$;
(2) $\langle \kappa_i : i < \eta\rangle$ converges to κ;
(3) $\kappa_{i+1}^{\kappa_i} = \kappa_{i+1}$ for all $i < \eta$.

Let θ be a large enough regular cardinal and fix an $\in$-increasing chain $\langle N_i \prec H(\theta) : i < \eta\rangle$ such that $|N_i| = \kappa_{i+1}$, $\kappa_{i+1} \subset N_i$, $\sup(N_i \cap \kappa_{i+2}) =_{def} \delta_i \in \kappa_{i+2} \cap cof(\kappa_{i+1})$, ${}^{\kappa_i} N_i \subset N_i$. We arrange that $\lambda, f, \langle \kappa_i : i < \eta\rangle \in N_0$.

We will recursively build $\langle A_i : i < \eta\rangle$ such that $A_i \subset N_i \cap \kappa_{i+2}$ and $|A_i| = \kappa_i^+$ satisfying:

for all $j \geq i$, $A_i \cup \{\delta_j\}$ is a rainbow subset of f.

Recursively, suppose $A_k \subset N_k \cap \kappa_{k+2}$ for $k < i$ have been built. Let $A^* = \bigcup_{k<i} A_k \subset \kappa_{i+1} \subset N_i$. Notice that $|A^*| \leq \kappa_i$. We will enlarge A^* with κ_i^+ many elements in $\delta_i - \kappa_{i+1}$. More precisely, we will find $C = \{\alpha_k \in \delta_i - \kappa_{i+1} : k < \kappa_i^+\}$ such that $A^* \cup C \cup \{\delta_j\}$ is a rainbow subset of f for all $j \geq i$. We finish by setting $A_i = A^* \cup C$.

Suppose we have built $C_\nu = \{\alpha_k : k < \nu\}$ for some $\nu < \kappa_i^+$ satisfying the requirement. Since ${}^{\kappa_i} N_i \subset N_i$, we have $A^* \cup C_\nu \in N_i$. Let $A(A^* \cup C_\nu) =_{def} \{\gamma < \kappa_{i+2} : A^* \cup C_\nu \cup \{\gamma\}$ is a rainbow subset for $f\}$. Since $A(A^* \cup C_\nu) \in N_i$ and $\delta_i \in A(A^* \cup C_\nu)$, we know that $A(A^* \cup C_\nu)$ is a stationary subset of κ_{i+2}.

Let $B_j =_{def} \{\rho \in A(A^* \cup C_\nu) : \exists \alpha \in A^* \cup C_\nu \ f(\alpha, \delta_j) = f(\rho, \delta_j)\}$ for each $j \geq i$. As $|A^* \cup C_\nu| \leq \kappa_i$ and the coloring is λ-bounded, we know that $|B_j| \leq \kappa_i$ for any $j \geq i$. Pick any $\gamma \in A(A^* \cup C_\nu) - \bigcup_{i \leq j < \eta} B_j$ with $\gamma > \max A^* \cup C_\nu$. We claim that this γ is as desired, namely $A^* \cup C_\nu \cup \{\gamma\} \cup \{\delta_j\}$ is a rainbow subset for all $j \geq i$. Indeed, fix some $j \geq i$. By the fact that $\gamma, \delta_j \in A(A^* \cup C_\nu)$, the only bad possibility is that for some $\alpha \in A^* \cup C_\nu$, $f(\alpha, \delta_j) = f(\gamma, \delta_j)$. But this is ruled out by the fact that $\gamma \notin B_j$.

$\square$

Remark 2.16. We can strengthen the conclusion of Lemma 2.14 to that $\kappa \to^{poly} (\kappa)^2_{\lambda-bdd}$ for any $\lambda < \kappa$ and it remains true in any forcing extension satisfying $< \gamma$-covering property (see Definition 3.11) for some cardinal $\gamma < \kappa$. The proof is similar to that of Theorem 3.1. Hence it is also possible for a singular cardinal which is not a strong limit to satisfy the conclusion of Lemma 2.14.

Remark 2.17. Combining the ideas from Theorem 2.7 and Lemma 2.14, we can show that: if λ is a regular cardinal and $\langle \kappa_i : i < \lambda\rangle$ is an increasing sequence of regular cardinals such that κ_i carries a κ_i-saturated κ_i-complete normal ideal for each $i < \lambda$, then $\kappa \to^{poly} (\kappa)^2_{\gamma-bdd}$ for all $\gamma < \kappa$, where $\kappa =_{def} \sup_{i<\lambda} \kappa_i$. Note that in this case the cardinal arithmetic assumptions as in Lemma 2.14 may not hold.

Remark 2.18. Lemma 2.14 provides a very sharp contrast: ZFC proves there exists an uncountable cardinal κ such that $\kappa \to^{poly} (\kappa)^2_{\lambda-bdd}$ for all $\lambda < \kappa$ while ZFC can not prove the existence of an uncountable cardinal κ satisfying $\kappa \to (\kappa)^2_2$.

Question 2.19. *If an inaccessible κ carries a non-trivial κ-complete κ-saturated normal ideal, is it true that $\kappa \to^{poly} (\kappa)^n_{\lambda-bdd}$ for all $n \in \omega$ and all $\lambda < \kappa$?*

3. The extent of Rainbow Ramsey theorems at successors of singular cardinals

In [2] and [1], it is shown that if GCH holds, then $\kappa^+ \to^{poly} (\eta)^2_{<\kappa-bdd}$ for any regular cardinal κ and ordinal $\eta < \kappa^+$ and moreover the partition relations continue to hold in any κ-c.c. forcing extension. The authors ask what we can say when κ is singular. We will address this question by showing $GCH + \square_\kappa$ implies

$\kappa^+ \to^{poly} (\eta)^2_{<\kappa-bdd}$ for all $\eta < cf(\kappa)^+$ and $\kappa^+ \not\to^{poly} (\eta)^2_{<\kappa-bdd}$ for all $\eta \geq cf(\kappa)^+$. As we will see below, a weaker hypothesis suffices.

Observation 3.1. If κ is singular of cofinality $\lambda < \kappa$, then $\kappa^+ \not\to^{poly} (\lambda^+ + 1)^2_{<\kappa-bdd}$.

Proof. For each $\beta \in \kappa^+$, fix disjoint $\{A_{\beta,n} : n \in \lambda\}$ such that each set has size $< \kappa$ and $\bigcup_{n\in\lambda} A_{\beta,n} = \beta$. Define a coloring by mapping $\{\alpha, \beta\} \in [\kappa^+]^2 \mapsto (n, \beta)$ if n is the unique element in λ that $\alpha \in A_{\beta,n}$. This coloring is easily seen to be $< \kappa$-bounded. For any subset A of order type $\lambda^+ + 1$, let δ be the top element. Now by pigeon hole, there exists $n \in \lambda$, such that $|A \cap A_{\delta,n}| \geq \lambda^+$. For any $\alpha < \beta \in A \cap A_{\delta,n}$, $f(\alpha, \delta) = (n, \delta) = f(\beta, \delta)$. Thus A is not a rainbow subset. □

Definition 3.2. Let κ be a cardinal of cofinality $\lambda < \kappa$. A *good covering matrix* on κ^+ is a collection $\{K_{\alpha,n} : \alpha < \kappa^+, n \in \lambda\}$ of subsets of κ^+ such that

1. $\bigcup_{n\in\omega} K_{\alpha,n} = \alpha$ for all $\alpha < \kappa^+$,
2. $|K_{\alpha,n}| < \kappa$ for all $\alpha < \kappa^+, n \in \lambda$,
3. for all $\alpha < \beta, n \in \lambda$, there is $m \in \lambda$ such that $K_{\alpha,n} \subset K_{\beta,m}$
4. for any $A \in [\kappa^+]^{\lambda^+}$, there exist $i < \lambda$, $\delta \in A$ and $A' \subset A \cap K_{\delta,i}$ with $|A'| \geq |i|^+$.

Lemma 3.3. *For singular κ with $\lambda = cf(\kappa) < \kappa$, if there exists a good covering matrix on κ^+, then $\kappa^+ \not\to^{poly} (\lambda^+)^2_{<\kappa-bdd}$*

Proof. Define f on $[\kappa^+]^2$ such that $f(\alpha, \beta) = (n, \beta)$ where n is the least $n \in \lambda$ such that $\alpha \in K_{\beta,n}$. Notice that this coloring is $< \kappa$-bounded since $|K_{\gamma,m}| < \kappa$ for all $\gamma \in \kappa^+, m \in \lambda$. Let $A \in [\kappa^+]^{\lambda^+}$. We claim that A is not a rainbow subset for f. By the property of a good covering matrix, there exists $i \in \lambda, \delta \in A, A' \subset K_{\delta,i} \cap A$ such that $|A'| = |i|^+$. By the definition of f, it is true that for each $\alpha \in A'$, $f(\alpha, \delta) \leq i$. By the Pigeonhole principle, there are $\alpha \neq \alpha' \in A'$ such that $f(\alpha, \delta) = f(\alpha', \delta)$. In particular, A is not a rainbow subset for f.

□

Definition 3.4 (Jensen matrix)**.** Let $\omega = cf(\kappa) < \kappa$. $\mathcal{D} = \langle K_{\alpha,i} : \alpha < \kappa^+, i < \omega\rangle$ is a *Jensen matrix* at κ^+ if

1. $|K_{\alpha,i}| < \kappa$ for all $\alpha < \kappa^+, i < \omega$,
2. for any $\alpha < \kappa^+$, $\alpha = \bigcup_{i<\omega} K_{\alpha,i}$,
3. for all $\alpha < \kappa^+$ and for all $i < j < \omega$, $K_{\alpha,i} \subset K_{\alpha,j}$,
4. for any $i < \omega$ and $\alpha < \beta < \kappa^+$, there is $j < \omega$ such that $K_{\alpha,i} \subset K_{\beta,j}$,
5. $\bigcup_{i\in\omega}[K_{\beta,i}]^\omega \subset \bigcup_{\alpha<\beta}\bigcup_{i\in\omega}[K_{\alpha,i}]^\omega$ whenever $cf(\beta) > \omega$.

Remark 3.5. In [9], Foreman and Magidor showed that the existence of a Jensen matrix at κ^+ is equivalent to a combinatorial principle called Very Weak Square at κ^+. They show this principle is consistent above a supercompact cardinal unlike $\Box_\kappa$ which must fail above any supercompact cardinal.

Lemma 3.6. *For singular cardinal κ of countable cofinality, any Jensen matrix at κ^+ is a good covering matrix at κ^+.*

Proof. We only need to verify the last requirement of a good covering matrix. Suppose $A \in [\kappa^+]^{\omega_1}$ is as given. Let $\gamma = \sup A$. Then there exists $i < \omega$ such that $A \cap K_{\gamma,i}$ is uncountable. Let A' be the first ω many elements of $A \cap K_{\gamma,i}$. Then there exists $\alpha < \gamma, j < \omega$ such that $A' \in [K_{\alpha,j}]^\omega$. Let $\beta \in A$ such that $\beta > \alpha$. By property (4) of a Jensen matrix, we can find $k \in \omega$ such that $K_{\alpha,j} \subset K_{\beta,k}$. Hence $A' \subset K_{\beta,k}$. Note that $|A'| \geq k+1$. The proof is finished. $\square$

The following connects the rainbow partition relations with sets in Shelah's approachability ideal. Fix a singular cardinal κ with cofinality λ.

Definition 3.7. A set $S \subset \kappa^+$ is in $I[\kappa^+;\kappa]$ iff there is a sequence $\bar{a} = \langle a_\alpha \in [\kappa^+]^{<\kappa} : \alpha < \lambda \rangle$ and a closed unbounded $C \subset \kappa^+$ such that for any $\delta \in C \cap S$ is singular and weakly approachable with respect to the sequence $\bar{a}$, namely there is an unbounded $A \subset \delta$ of order type $cf(\delta)$ such that any $\alpha < \delta$ there exists $\beta < \delta$ with $A \cap \alpha \subset a_\beta$.

Notice that $I[\kappa^+;\kappa]$ contains $I[\kappa^+]$, which is Shelah's approachability ideal. For more details on these matters, see [7].

Definition 3.8 (Definition 3.24, 3.25 [7])**.** $d : [\kappa^+]^2 \to cf(\kappa)$ is

(1) *normal* if
$$i < cf(\kappa) \to \sup_{\alpha < \kappa^+} |\{\beta < \alpha : d(\beta, \alpha) < i\}| < \kappa,$$
(2) *transitive* if for any $\alpha < \gamma < \beta < \kappa^+$, $d(\alpha,\beta) \leq \max\{d(\alpha,\gamma), d(\gamma,\beta)\}$,

(3) *approachable* at $S \subset \lim \kappa^+$ if for any $\delta \in S$, there is a cofinal $A \subset \delta$ such that for any $\alpha \in A$, $\sup\{d(\beta,\alpha) : \beta \in A \cap \alpha\} < cf(\kappa)$.

It is a consequence of Theorem 3.28 in [7] that $\kappa^+ \cap cof(\lambda^+) \in I[\kappa^+;\kappa]$ implies the existence of a normal d that is approachable at $E \cap \kappa^+ \cap cof(\lambda^+)$ for some club $E \subset \kappa^+$.

Claim 3.9. *$\kappa^+ \cap cof(\lambda^+) \in I[\kappa^+;\kappa]$ implies $\kappa^+ \not\to^{poly} (\lambda^+)^2_{<\kappa-bdd}$.*

Proof. Fix a normal d that is approachable at $E \cap \kappa^+ \cap cof(\lambda^+)$ for some club $E \subset \kappa^+$. Define $f : [E]^2 \to \kappa^+$ such that $f(\alpha,\beta) = (d(\alpha,\beta),\beta)$. The normality of d implies f is $<\kappa$-bounded. Given $A \in [E]^{\lambda^+}$, let $\gamma = \sup A$. Then d is approachable at γ. Fix some unbounded $B \subset \gamma$ of order type λ^+ witnessing the approachability of d. We may assume there exists $\eta_0 < \lambda$ such that $\sup d''[B]^2 \leq \eta_0$. Pick the following increasing sequences $\langle a_i \in A : i < \lambda^+ \rangle$ and $\langle b_i \in B : i < \lambda^+ \rangle$ satisfying the for all $i < \lambda^+$, $b_i < a_i < b_{i+1}$. By the Pigeon Hole principle, we can find $D \in [\lambda^+]^{\lambda^+}$ and some $\eta_1 \in \lambda$ such that for all $i \in D$, $d(b_i,a_i), d(a_i,b_{i+1}) \leq \eta_1$. Then for any $i < j \in D$, by the transitivity of d, we have $d(a_i,a_j) \leq \max\{d(a_i,b_{i+1}), d(b_{i+1},b_j), d(b_j,a_j)\} \leq \max\{\eta_0,\eta_1\} =_{def} \eta^*$ (here we use the convention that $d(t,t) = 0$). Let $A' = \{a_i : i \in D\}$.

Pick $\delta \in A'$ such that $A' \cap \delta$ has size λ. We know $\sup d(\cdot,\delta)'' A' \cap \delta \leq \eta^* < \lambda$, which clearly implies there exist $\alpha_0 < \alpha_1 \in A' \cap \delta$ such that $d(\alpha_0,\delta) = d(\alpha_1,\delta)$ so in particular A is not rainbow. $\square$

Remark 3.10. $\square_\kappa$ implies $I[\kappa^+]$ is trivial. Hence $\square_\kappa$ implies $\kappa^+ \not\to^{poly} (cf(\kappa)^+)^2_{<\kappa-bdd}$.

In light of the preceding theorems, the following theorem is the best possible in a sense. Recall κ is a singular cardinal with cofinality λ.

Definition 3.11. A forcing poset $\mathbb{P}$ satisfies $<\kappa$-covering property if for any $\mathbb{P}$-name of subset of ordinals $\dot{B}$ such that $\Vdash_{\mathbb{P}} |\dot{B}| < \kappa$, there exists $B \in V$ such that $|B| < \kappa$ and $\Vdash_{\mathbb{P}} \dot{B} \subset B$.

Notice that κ and κ^+ are preserved as cardinals in any forcing extension satisfying $<\kappa$-covering property.

Theorem 3.1. *Suppose $\kappa^{<\lambda} = \kappa$. Then for any $\alpha < \lambda^+$,*

$$\kappa^+ \rightarrow^{poly} (\alpha)^2_{<\kappa-bdd}. \tag{3.12}$$

Moreover, these partition relations continue to hold in any forcing extension by $\mathbb{P}$ satisfying the $<\kappa$-covering property.

Proof. We may assume $|\alpha| = \lambda$. Fix a $\mathbb{P}$-name for a $<\kappa$-bounded coloring $\dot{f}$ on $[\kappa^+]^2$. We may assume it is normal. Fix some large enough regular cardinal χ. Build a sequence $\langle M_i \prec (H(\chi), \in, \dot{f}, \kappa, \mathbb{P}) : i < \alpha\rangle$ such that

(1) $\kappa + 1 \subset M_i$, $|M_i| = \kappa$, $\kappa_i =_{def} M_i \cap \kappa^+ \in \kappa^+$,
(2) $|\kappa_{i+1} - \kappa_i| = \kappa$,
(3) ${}^{<\lambda}M_i \subset M_{i+1}$.

The construction is possible since $\kappa^{<\lambda} = \kappa$. Fix a bijection $g : \lambda \to \alpha$. We will inductively define a rainbow subset $\{a_i : i < \lambda\}$ such that $a_i \in \kappa_{g(i)+1} - \kappa_{g(i)}$. It is clear that this set as defined will have order type α. During the construction, we maintain the following *construction invariant*:

for any $i < \lambda$ and $l = g(i)$, whenever $a_j, a_k < \kappa_{l+1}$, we have $\Vdash_{\mathbb{P}} \dot{f}(a_j, \kappa_{l+1}) \neq \dot{f}(a_k, \kappa_{l+1})$.

Suppose for some $\beta < \lambda$ we have defined $A = \{a_i : i < \beta\}$. Let $l = g(\beta)$ and $B = \kappa_{l+1} - \kappa_l$. Our goal is to find an element in B such that after we augment A with this element, not only does the set remains a rainbow subset, but also the construction invariant is satisfied. Let $C = \{\delta < \kappa^+ : \forall i, j < \beta \ a_i, a_j \in A \cap \kappa_{l+1} \rightarrow \Vdash_{\mathbb{P}} \dot{f}(a_i, \delta) \neq \dot{f}(a_j, \delta)\}$ and $B' = B \cap C$.

Claim 3.13. $|B'| = \kappa$.

Proof of the claim. Let $A' = A \cap M_{l+1} = A \cap \kappa_{l+1} \subset M_l$. As ${}^{<\lambda}M_l \subset M_{l+1}$ we have $A' \in M_{l+1}$. Hence $C \in M_{l+1}$ and that $\kappa_{l+1} \in C$ by the construction invariant. C is thus a stationary subset of κ^+. In particular, $M_{l+1} \models$ there exists an injection from κ to C. As $\kappa + 1 \subset M_{l+1}$, $B \cap C = B'$ has size κ. $\square$

We want to pick an element from B' and add it to the set, however, we need to make sure the set is rainbow and satisfy the construction invariant. For any cardinal δ, let $A \restriction \delta$ be $A \cap (<\delta)$. For the purpose of presentation, work in $V[G]$ for some $G \subset \mathbb{P}$ generic over V.

Let $B_{-1} = \{\delta \in B' : \exists a \in A \restriction \kappa_{l+1} \ f(\delta, \kappa_{l+1}) = f(a, \kappa_{l+1})\}$. For each $i < \beta$ with $g(i) > l$, let $B_i = \{\delta \in B' : \exists \alpha \in A \restriction \kappa_{g(i)+1} \ f(\alpha, \kappa_{g(i)+1}) = f(\delta, \kappa_{g(i)+1})\}$ and $B'_i = \{\delta \in B' : \exists \alpha \in A \restriction a_i \ f(\alpha, a_i) = f(\delta, a_i)\}$. We verify that these sets as defined all have size $<\kappa$.

Suppose for the sake of contradiction that B_{-1} has size κ, then since $|A| < \kappa$ and $|B'| = \kappa$, there exists $a \in A$ such that $\{\delta \in B' : f(a, \kappa_{l+1}) = f(\delta, \kappa_{l+1})\}$ has size κ. This contradicts with the assumption that f is $<\kappa$-bounded.

Suppose for the sake of contradiction that for some i with $i < \beta$ and $g(i) > l$ we have $|B_i| = \kappa$, similar to the above, we can find $a \in A$ such that $\{\delta \in B' : f(a, \kappa_{g(i)+1}) = f(\delta, \kappa_{g(i)+1})\}$ has size κ, contradicting with $<\kappa$-boundedness. Similarly $|B'_i| < \kappa$.

Back in V, pick $\mathbb{P}$-names for the sets above: $\dot{B}_{-1}$, $\dot{B}_i, \dot{B}'_i$ for all $i < \beta$ such that $g(i) > l$. By the $<\kappa$-covering property of $\mathbb{P}$, we can find $B^*_{-1}, B^*_i, (B'_i)^*$ of size $< \kappa$ in V such that $\Vdash_{\mathbb{P}} \dot{B}_{-1} \subset B^*_{-1}, \dot{B}_i \subset B^*_i, \dot{B}'_i \subset (B'_i)^*$ for all $i < \beta$ with $g(i) > l$. Since $\beta < \lambda = cf(\kappa)$, we know $|B^*_{-1} \cup \bigcup_{i<\beta, g(i)>l} B^*_i \cup (B'_i)^*| < \kappa$. Pick $a_\beta \in B' - B^*_{-1} - \bigcup_{i<\beta, g(i)>l} B^*_i \cup (B'_i)^*$. Then it follows that $A \cup \{a_\beta\}$ is forced by $\mathbb{P}$ to be a rainbow subset and to satisfy the construction invariant.

□

An immediate consequence of the proof of Theorem 3.1 is:

Corollary 3.14. *For any cardinal κ and any $\alpha < \omega_1$,*

$$\kappa^+ \to^{poly} (\alpha)^2_{<\kappa-bdd}. \tag{3.15}$$

Question 3.16. *Is $\kappa^+ \to^{poly} (\omega_1)^2_{<\kappa-bdd}$ consistent for some singular κ of countable cofinality?*

4. A coloring that is strongly proper indestructible but c.c.c destructible

It is proved in [2] that if CH holds, then $\omega_2 \to^{poly} (\eta)^2_{<\omega_1-bdd}$ for any $\eta < \omega_2$. In [1], a model where $2^\omega = \omega_2$ and $\omega_2 \not\to^{poly} (\omega_1)^2_{2-bdd}$ is constructed. A question regarding the possibility of getting $\omega_2 \not\to^{poly} (\omega_1)^2_{2-bdd}$ along with continuum larger than ω_2 was raised. A positive answer was given in [4] using the method of forcing with symmetric systems of submodels as side conditions.

In this section we give a simplified construction of the model presented in [1] using the framework developed by Neeman [11] and show the witness to $\omega_2 \not\to^{poly} (\omega_1)^2_{2-bdd}$ in that model is indestructible under strongly proper forcings. This provides an alternative answer to the original question.

Definition 4.1 (Special case of Definition 2.2 and 2.4 in [11])**.** Let $K = (H(\omega_2), <^*)$ where $<^*$ is some well-ordering of $H(\omega_2)$. Define *small nodes* and *transitive nodes* respectively as

$$\mathcal{S} =_{def} \{M \in [K]^\omega : M \prec K\}$$

and

$$\mathcal{T} =_{def} \{W \in [K]^{\omega_1} : W \prec K \text{ and internally approachable of length } \omega_1\}.$$

Both sets are stationary in K respectively. $\mathbb{P} = \mathbb{P}_{\omega,\omega_1,\mathcal{S},\mathcal{T}}$ is the standard sequence poset consisting of models of two types. More precisely, $\mathbb{P}$ consists of finite increasing $\in$-chain of elements in $\mathcal{S} \cup \mathcal{T}$ closed under intersection.

We will assume familiarity of some basic properties of $\mathbb{P}$. It will be helpful to have a copy of [11] at hand but we will list the lemmas needed.

Claim 4.2 (Claim 2.17, 2.18)**.** *Fix $s \in \mathbb{P}$ and $Q \in s$. Define $res_Q(s) = s \cap Q$. Then*

(1) $res_Q(s) \in \mathbb{P}$.
(2) If Q is a transitive node, then $res_Q(s)$ consists of all nodes of s that occur before Q. If Q is a small node, then $res_Q(s)$ consists of all nodes in s that occur before Q and do not belong to any interval $[Q \cap W, W) \cap s$ for any transitive node $W \in s$. Those intervals are called residue gaps *of s in Q.*

Lemma 4.3 (Corollary 2.31 in [11])**.** *Let $s \in \mathbb{P}$ and $Q \in s$. For any $t \in \mathbb{P} \cap Q$ such that $t \leq res_Q(s) =_{def} s \cap Q \in \mathbb{P}$. Then*

(1) s and t are directly compatible, namely the closure of $s \cup t$ under intersection is a common lower bound. Moreover, if Q is a transitive node, then $s \cup t$ is already closed under intersection hence is the lower bound for s and t.
(2) If r is the closure of $s \cup t$, then $res_Q(r) = t$.
(3) The small nodes outside Q are of the form N or $N \cap W$ where N is a small node of s and W is a transitive node of t.

For each $\beta < \omega_2$, let f_β be the $<^*$-least injection from β to ω_1. Define the main forcing $\mathbb{Q}$ to consist of $p = (c_p, s_p)$ such that:

(1) c_p is a finite partial function from $[\omega_2]^2 \to \omega_1$ satisfying the rainbow requirement, namely there do not exist $\alpha_0 < \alpha_1 < \alpha_2 < \beta$ such that $(\alpha_i, \beta) \in dom(c_p)$ for all $i < 3$ and $c_p(\alpha_0, \beta) = c_p(\alpha_1, \beta) = c_p(\alpha_2, \beta)$;
(2) for any $(\alpha, \beta) \in dom(c_p)$, $c_p(\alpha, \beta) \geq f_\beta(\alpha)$;
(3) $s_p \in \mathbb{P}$

$q \leq p$ iff $c_q \restriction dom(c_p) = c_p$ and $s_q \supset s_p$ and for any $(\alpha, \beta) \in dom(c_q) \backslash dom(c_p)$ and $M \in s_p$, if $(\alpha, \beta) \in M$ then $c_q(\alpha, \beta) \in M$.

Claim 4.4. *For any $\alpha < \beta < \omega_2$ and $p \in \mathbb{Q}$, there exists $p' \leq p$ such that $(\alpha, \beta) \in dom(c_{p'})$.*

Proof. We may assume $(\alpha, \beta) \notin dom(c_p)$. Consider $A = \{M \in s_p : (\alpha, \beta) \in M\}$. As the nodes are closed under intersection, we know that $\bigcap A = M_0 \in s_p$, which is minimal under $\subset$. Since $(\alpha, \beta) \in M_0$, $f_\beta(\alpha) \in M_0$. Pick $\gamma \in M_0 \cap \omega_1 \backslash (f_\beta(\alpha) + 1)$ which is not in $range(c_p)$. It is clear that $(c_p \cup (\{\alpha, \beta\}, \gamma), s_p)$ is a desired extension. $\square$

Definition 4.5. Let λ be a fixed regular cardinal, P be a poset. Let $\mathcal{M} = (H(\lambda), \in, \cdots)$ be some countable extension of $(H(\lambda), \in)$. We say P is strongly proper for B where $B \subset \{M : M \prec \mathcal{M}\}$ if for any $M \in B$ and any $r \in M \cap P$, there exists $r' \leq r$ such that r' is *strongly (M, P)-generic*, namely for any $r'' \leq r'$, there exists a *reduct* $r'' \restriction M \in M \cap P$ and $r'' \restriction M \geq r''$ such that any $t \leq r'' \restriction M$ with $t \in M$ is compatible with r''.

P is strongly proper if for all sufficiently large θ, P is strongly proper for a club subset of $\{M \in [H(\theta)]^\omega : M \prec H(\theta)\}$.

Claim 4.6. *For any $p = (c_p, s_p)$ with a transitive node $W \in s_p$, if $t \leq (c_p \cap W, res_W(s_p))$ and $t \in W$, then t and p are compatible. Hence $\mathbb{Q}$ is strongly proper for $\mathcal{T}$.*

Proof. Implicitly in the statement of the claim, $(c_p \cap W, res_W(s_p))$ can be easily checked to be a condition. Note that $r = (c_t \cup c_p, s_p \cup s_t)$ is a condition and by Lemma 4.3, $s_p \cup s_t \leq_{\mathbb{P}} s_p, s_t$. We want to show this condition extends both t and p. To see $r \leq t$, for any $(\alpha, \beta) \in dom(c_r) - dom(c_t)$, $(\alpha, \beta) \notin W$ so $(\alpha, \beta) \notin M$ for any $M \in s_t$ as $t \in W$ and W is transitive. To see $r \leq p$, for any $(\alpha, \beta) \in dom(c_r) - dom(c_p) = dom(c_t) - dom(c_p)$, if $M \in s_p \cap W$ such that $(\alpha, \beta) \in M \subset W$, then $c_t(\alpha, \beta) = c_r(\alpha, \beta) \in M$ by the fact that $t \leq (c_p \cap W, res_W(s_p))$. For $M \notin W$, since $M \cap W \in W \cap s_p$, by the argument before we know $c_p(\alpha, \beta) \in M \cap W \subset M$.

To see $\mathbb{Q}$ is strongly proper for $\mathcal{T}$, it suffices to notice that for any $W \in \mathcal{T}$ and $t = (c_t, s_t) \in W \cap \mathbb{Q}$, there exists $t' = (c_t, s'_t) \leq t$ such that $W \in s'_t$ by Lemma 4.3. $\square$

Claim 4.7. *For any countable $M^* \prec H(\lambda)$ for some large enough regular λ containing $\mathbb{Q}, K$, any $(c_p, s_p) \in M^* \cap \mathbb{Q}$ extends to a strongly $(M^*, \mathbb{Q})$-generic condition, $r = (c_p, s_p \cup \{M^* \cap K\})$. In particular, $\mathbb{Q}$ is strongly proper.*

Proof. Let $M = M^* \cap K$. We show for any $r' \leq r$, there exists $r' \restriction M \geq r$ and $r' \restriction M \in M$, such that any extension of $r' \restriction M$ in M is compatible with r'.

First note that for any $(\alpha, \beta) \in M$, $c_{r'}(\alpha, \beta) \in M$. If $(\alpha, \beta) \in dom(c_p)$, then it is true as $p \in M$. If $(\alpha, \beta) \notin dom(c_p)$, by the extension requirement and the fact that $M \in s_r$, we know that $c_{r'}(\alpha, \beta) \in M$. Let $r' \restriction M$ be $(c_{r'} \cap M, res_M(s_{r'}))$. It is easy to see that $r' \restriction M$ is a condition. To see $r' \leq r' \restriction M$, we only need to note that for any $(\alpha, \beta) \in dom(c_{r'}) - dom(c_{r'} \cap M)$ and $N \in res_M(r') \cap \mathcal{S}$, $(\alpha, \beta) \notin N$ since $N \subset M$.

Let $t \in \mathbb{Q} \cap M$ be such that such that $t \leq r' \restriction M$. As $s_t \leq res_M(s_{r'})$ and $s_t \in M$, we know by Lemma 4.3 there exists $s^* \leq s_t, s_{r'}$ such that $res_M(s^*) = s_t$ and any small node of s^* outside M is either a small node in $s_{r'}$ or of the form $N \cap W$ where N is a small node in $s_{r'}$ and W is a transitive node in s_t.

If we manage to show $(c_t \cup c_{r'}, s^*)$ is a condition that extends both t and r' then we are done.

First we check that $h = (c_t \cup c_{r'}, s^*)$ is a condition. We will only verify the rainbow requirement, namely $c_t \cup c_{r'}$ is a partial function that is 2-bounded. Other requirements are straightforward to verify. To see it is a function, let $(\alpha, \beta) \in dom(c_t) \cap dom(c_{r'})$, then $(\alpha, \beta) \in M$. Since $c_t \supset c_{r'} \restriction M$, we know $c_t(\alpha, \beta) = c_{r'}(\alpha, \beta)$. To see $c_t \cup c_{r'}$ is 2-bounded, suppose for the sake of contradiction, $\alpha_0 < \alpha_1 < \alpha_2 < \beta$ are such that $c_h(\alpha_0, \beta) = c_h(\alpha_1, \beta) = c_h(\alpha_2, \beta) = \gamma \in \omega_1$. Note that there exists some $i < 3$ such that $(\alpha_i, \beta) \in M$ since otherwise $(\alpha_k, \beta) \in dom(c_{r'})$ for all $k < 3$, which contradicts with the fact that r' is a condition. Also notice that $c_t(\alpha_i, \beta) = \gamma \in M$. By the requirement of a condition we know $f_\beta(\alpha_j) \leq \gamma$ for all $j < 3$. But as $\gamma \in M$, $\gamma \subset M$, we know $\alpha_j \in M$ for all $j < 3$. This means these three tuples are all in the domain of c_t. This is a contradiction to the fact that t is a condition.

Finally we check that $h \leq t, r'$. To see $h \leq t$, fix some $(\alpha, \beta) \in dom(c_h) - dom(c_t) = dom(c_{r'}) - dom(c_t)$ and $N \in s_t \cap \mathcal{S}$. Since $t \in M$, $N \in M$ hence $N \subset M$. But $(\alpha, \beta) \notin M$, so the requirement is satisfied vacuously. To see $h \leq r'$, fix some $(\alpha, \beta) \in dom(c_t) - dom(c_{r'})$ and $N \in s_{r'} \cap \mathcal{S}$ such that $(\alpha, \beta) \in N$. Since $(\alpha, \beta) \in M$ and $s_{r'}$ is closed under intersection, we may assume $N \subset M$. If $N = M$, then we are done since $c_h(\alpha, \beta) = c_{r'}(\alpha, \beta) \in M$. If $N \in M$, then we are done since $t \leq r' \restriction M$. So assume $N \notin M$. By Claim 4.2, N occurs in a residue gap, namely there exists $W \in M$ such that $N \in [W \cap M, W) =_{def} \{M' \in s_{r'} : rank(W \cap M) \leq rank(M') < rank(W)\}$. We will show $c_h(\alpha, \beta) \in N$ by inducting on the rank of the associated W. As $(\alpha, \beta) \in N \subset W$, $(\alpha, \beta) \in M \cap W$. Also $c_h(\alpha, \beta) \in M \cap W$. If there is no transitive node between $W \cap M$ and N, then we are done since $W \cap M \subset N$ (recall that $s_{r'}$ is linearly ordered by $\in$). Otherwise, there exists $W' \in [W \cap M, W) \cap \mathcal{T}$ whose rank is the least. Let $N' = W' \cap N$. Notice that $(\alpha, \beta) \in N'$. If $N' \in [W \cap M, W)$, then we are done as before as the minimality of W' ensures there is no transitive node between N' and $W \cap M$. Otherwise, if $N' \in M$, then the conclusion holds as before. If $N' \notin M$, then it lies in some residue gap $[W^* \cap M, W^*)$ and furthermore, $rank(W^*) < rank(W)$. By the induction hypothesis, we know $c_h(\alpha, \beta) \in N' \subset N$.

$\square$

By Claim 4.7 and Claim 4.6, ω_1 and ω_2 are preserved in the forcing extension by $\mathbb{Q}$.

Lemma 4.8 (Lemma 4.3 of [1])**.** *For $\alpha_0 < \alpha_1 < \beta < \omega_2$ and $p \in \mathbb{Q}$, if $(\alpha_i, \beta) \notin dom(c_p)$ for any $i < 2$ and*

$$\forall M \in s_p \ (\alpha_0, \beta) \in M \Leftrightarrow (\alpha_1, \beta) \in M \tag{4.9}$$

Then there exists an extension $p' = (c_{p'}, s_p)$ such that $(\alpha_0, \beta), (\alpha_1, \beta) \in dom(c_{p'})$ and $c_{p'}(\alpha_0, \beta) = c_{p'}(\alpha_1, \beta)$. Furthermore, we can ensure that $dom(c_{p'}) = dom(c_p) \cup \{(\alpha_0, \beta), (\alpha_1, \beta)\}$.

Building on the idea of Lemma 4.6 in [1], we prove a strengthened version in the following.

Lemma 4.10. *In $V^{\mathbb{Q}}$, for any strongly proper forcing $\dot{P}$, $\Vdash_{\dot{P}}$ c witnesses $\omega_2^V \not\to^{poly} (\omega_1)^2_{2-bdd}$.*

Remark 4.11. More accurately, it is the coloring $c' : [\omega_2]^2 \to \omega_2$ such that $c'(\alpha, \beta) = (c(\alpha, \beta), \beta)$ that witnesses $\omega_2 \not\to^{poly} (\omega_1)^2_{2-bdd}$. As it is clear from the context, we will continue to refer to c as the witness in the following.

Proof of Lemma 4.10. Suppose otherwise for the sake of contradiction. Let $r \in \mathbb{Q}$, $\mathbb{Q}$-name $\dot{p}, \dot{P}$, $\mathbb{Q} * \dot{P}$-name $\dot{X}$, $\gamma \in \omega_2^V + 1$ such that

1. $r \Vdash_{\mathbb{Q}} \dot{P}$ is a strongly proper forcing and $\dot{p} \in \dot{P}$ and
2. $r \Vdash_{\mathbb{Q}} \dot{p} \Vdash_{\dot{P}} \sup \dot{X} = \gamma$, $\dot{X}$ is a rainbow subset for c of order type ω_1.

Note that we include the possibility that $\gamma = \omega_2^V$ since it may be collapsed by $\mathbb{Q} * \dot{P}$. In either case, $cf(\gamma) > \omega$.

Let $G \subset \mathbb{Q}$ containing r be generic over V. Fix some sufficiently large regular cardinal λ and let $C = (\dot{C})^G \subset ([H(\lambda)]^\omega)^{V[G]}$ be a club that witnesses the strong properness of P in $V[G]$.

Claim 4.12. *For any stationary subset $T \subset [H(\lambda)]^\omega$ in V, $T[G] =_{def} \{M[G] : M \in T\}$ is a stationary subset of $([H(\lambda)]^\omega)^{V[G]}$.*

Proof of the claim. In $V[G]$, let $f : H(\lambda)^{<\omega} \to H(\lambda)$. In V, let λ^* be much larger regular cardinal than λ and $M' \prec H(\lambda^*)$ containing $\dot{f}, H(\lambda)$ be such that $M = M' \cap H(\lambda) \in T$. Then $M[G]$ is closed under f, since for any $\bar{a} \in M[G] \cap [H(\lambda)^{V[G]}]^{<\omega}$, $f(a) \in M'[G] \cap (H(\lambda))^{V[G]} = M'[G] \cap H(\lambda)[G] = (M' \cap H(\lambda))[G]$. The last equality holds since for any $\dot{\tau} \in M', \dot{\sigma} \in H(\lambda)$ such that $(\dot{\tau})^G = (\dot{\sigma})^G$, by the fact that $M'[G] \prec H(\lambda^*)[G]$, $M'[G] \models$ there exists $\dot{\sigma} \in H(\lambda)^V$, $\dot{\tau}^G = \dot{\sigma}^G$. It is easy to see this is sufficient since $M'[G] \cap H(\lambda)^V = M' \cap H(\lambda)^V$. □

Find a countable $N' \in V$ such that $N' \prec H(\lambda)^V$ contains $r, \mathbb{Q}, \dot{p}, \dot{P}, \dot{X}, \gamma$. Moreover, $N =_{def} N' \cap K \in \mathcal{S}$ and $N'[G] \in C$.

Let $\gamma' = \sup N \cap \gamma$. Extend r to t such that $N \in s_t$ by Lemma 4.3. Consequently, t is strongly $(N', \mathbb{Q})$-generic. Find $t' \leq_{\mathbb{Q}} t$, $\beta \in [\gamma', \gamma)$ and $\mathbb{Q}$-names $\dot{p}'$, $\dot{q}$ such that $\dot{q} \in N'$ and $t' \Vdash_{\mathbb{Q}} \dot{p}'$ is strongly $(N'[\dot{G}], \dot{P})$-generic and $\dot{p}' \leq_{\dot{P}} \dot{p}$, $\dot{p}' \restriction N'[\dot{G}] = \dot{q}$ and $\dot{p}' \Vdash_{\dot{P}} \beta \in \dot{X}$. Let $m = |t'| < \omega$.

Now consider $D = \{a \leq_{\mathbb{Q}} t' \restriction N' : \exists \dot{b} \ a \Vdash_{\mathbb{Q}} \dot{b} \leq_{\dot{P}} \dot{q}, \exists \alpha_0 < \cdots < \alpha_{2^m} \ \dot{b} \Vdash_{\dot{P}} \forall i \leq 2^m \ \alpha_i \in \dot{X}\}$. This set is dense below $t' \restriction N'$ and is in N'. Pick $a \in D \cap N'$ and $\dot{b}, \alpha_0, \cdots, \alpha_{2^m} \in N'$ as its witness. By the Pigeonhole principle, there exist

$i \neq j \leq 2^m$ such that for any $M \in \mathcal{S} \cap s_{t'}$, $(\alpha_i, \beta) \in M$ iff $(\alpha_j, \beta) \in M$. Apply Lemma 4.8, there exists $t'' \leq t'$ such that $c_{t''}(\alpha_i, \beta) = c_{t''}(\alpha_j, \beta)$ with $s_{t''} = s_{t'}$ and $dom(c_{t''}) = dom(c_{t'}) \cup \{(\alpha_i, \beta), (\alpha_j, \beta)\}$. As $a \leq_{\mathbb{Q}} t' \restriction N' = t'' \restriction N'$, a and t'' are compatible. Find a common lower bound $t''' \leq_{\mathbb{Q}} a, t''$. Then $t''' \Vdash_{\mathbb{Q}} \dot{b} \leq_{\dot{P}} \dot{q} = \dot{p}' \restriction N'[\dot{G}]$ and $\dot{b} \in N'[\dot{G}]$. Hence t''' forces $\dot{b}$ and $\dot{p}'$ are compatible. Let $\dot{w}$ be a common lower bound. Then $(t''', \dot{w})$ forces $c(\alpha_i, \beta) = c(\alpha_j, \beta)$ as well as $\alpha_i, \alpha_j, \beta \in \dot{X}$. This is a contradiction since $(t''', \dot{w}) \leq_{\mathbb{Q} * \dot{P}} (r, \dot{p})$ and $(r, \dot{p}) \Vdash_{\mathbb{Q} * \dot{P}} \dot{X}$ is a rainbow subset for c.

□

An immediate consequence is $\omega_2 \not\to^{poly} (\omega_1)^2_{2-bdd}$ is consistent with the continuum being arbitrarily large as Cohen forcings are strongly proper. This provides an alternative answer to a question in [1], which was originally answered in [4] using a different method.

However In this model, there exists a c.c.c forcing that forces a rainbow subset into $c \restriction [\omega_1]^2$. In $V^{\mathbb{Q}}$, let R be the poset $\{a \in [\omega_1]^{<\omega} : a \text{ is a rainbow subset for } c\}$ order by reverse inclusion. By Remark 4.11, $a \in [\omega_1]^{<\omega}$ is a rainbow subset for c if there is no $\alpha_0 < \alpha_1 < \beta \in c$ such that $c(\alpha_0, \beta) = c(\alpha_1, \beta)$. It is easy to see that in $V^{\mathbb{Q}}$, R adds an unbounded subset of ω_1^V.

Lemma 4.13. *In $V^{\mathbb{Q}}$, R is c.c.c.*

Proof. Otherwise, let $\langle \dot{\tau}_i : i < \omega \rangle$ be a head-tail-tail system with root $r \in [\omega_1]^{<\omega}$ that is forced to be an uncountable antichain by p. Let $N' \prec H(\lambda)$ contain relevant objects for some sufficiently large regular cardinal λ. Let $\delta = N' \cap \omega_1$. Let $q \leq p$ be a strongly $(N', \mathbb{Q})$-generic condition that determines some $\dot{\tau}_j = h$ such that $\min(h - r) \geq \delta$. Let $q' = q \restriction N'$. Find $t \leq q'$ in N' such that t decides some $\dot{\tau}_i = h' \in N'$ such that $\min(h' - r) \geq \max_{(\alpha,\beta) \in dom(c_q) \cap N'} \max\{\alpha, \beta\} + 1$. Now we extend q to q^* such that $s_q = s_{q^*}$ and $dom(c_{q^*})$ includes $h' \times h$ such that $c_{q^*}[(h' - r) \times (h - r)] \cap (\delta \cup range(c_q)) = \emptyset$, $c_{q^*} \restriction (h' - r) \times (h - r)$ is injective and $q^* \restriction N' = q'$. To see that we can do this, enumerate $(h' - r) \times (h - r)$ as $\{(\alpha_i, \beta_i) : i < k\}$. We inductively add (α_i, β_i) to c_q by Claim 4.4 while maintaining the other requirements. More precisely, suppose we have added (α_j, β_j) to the domain of c_p for $j < i$. Let $M \in s_p$ be of the minimum rank such that $(\alpha_i, \beta_i) \in M$. Then $M \cap \omega_1 > \max\{\delta, f_{\beta_i}(\alpha_i)\}$. Hence we only need to avoid finitely many elements in $M \cap \omega_1 - (\max\{\delta, f_{\beta_i}(\alpha_i)\} + 1)$, which is clearly possible. q^* is compatible with t since $t \leq q' = q^* \restriction N'$ and $q^* \leq q$ which is strongly $(N', \mathbb{Q})$-generic. But a common extension of q^* and t forces that $\dot{\tau}_i \cup \dot{\tau}_j$ is rainbow. We have reached the desired contradiction. □

5. Some remarks and questions on partition relations of triples

Recall that Todorcevic in [14] showed that it is consistent that $\omega_1 \to^{poly} (\omega_1)^2_{<\omega-bdd}$. In fact, he showed a stronger conclusion, namely for any $<\omega$-bounded coloring on $[\omega_1]^2$, it is always possible to partition ω_1 into countably many rainbow subsets. He also showed the conclusion follows from PFA.

The plain generalization of this result to 3-dimensional case fails miserably.

Remark 5.1. $\omega_1 \not\to^{poly} (4)^3_{<\omega-bdd}$.

Proof. Fix $a : [\omega_1]^2 \to \omega$ such that for each $\alpha < \omega_1$, $a(\cdot, \alpha)$ is an injection from α to ω. Define $f : [\omega_1]^3 \to \omega$ such that $\{\alpha, \beta, \gamma\}_<$ is defined to be $\max\{a(\alpha, \gamma), a(\beta, \gamma)\} \in$

ω. Now define $g : [\omega_1]^3 \to \omega_1$ to be $g(\{\alpha, \beta, \gamma\}) = (f(\{\alpha, \beta, \gamma\}), \gamma)$. Note g is $< \omega$-bounded, since for each $\gamma \in \omega$, there are only finitely many $\alpha < \gamma$ such that $a(\alpha, \gamma) < n$. For any $A = \{\alpha_0 < \alpha_1 < \alpha_2 < \alpha_3\} \subset \omega_1$ of size 4, pick $i < 3$ such that for any $j < 3$ and $j \neq i$, $a(\alpha_j, \alpha_3) < a(\alpha_i, \alpha_3) = n$. Say $i = 0$ for the sake of demonstration. Then $\{\alpha_0, \alpha_1, \alpha_3\}$ and $\{\alpha_0, \alpha_2, \alpha_3\}$ get the same color (n, γ). $\square$

Remark 5.2. There are various limitations on Ramsey Theorems for higher dimensions. For example, $2^\omega \not\to (\omega + 2)^3_2$. Hence we need other methods to prove higher dimensional rainbow Ramsey theorems.

Given a 2-bounded normal coloring f on $[\delta]^3$, let us try to classify what types of obstacles there are for getting a rainbow subset.

Type 1 for some $\alpha, \beta, \alpha', \beta' < \gamma$ such that $\{\alpha, \beta\} \cap \{\alpha', \beta'\} = \emptyset$ and $f(\alpha, \beta, \gamma) = f(\alpha', \beta', \gamma)$

Type 2 for some $\alpha < \beta < \gamma < \delta$, $f(\alpha, \gamma, \delta) = f(\alpha, \beta, \delta)$

Type 3 for some $\alpha < \beta < \gamma < \delta$, $f(\alpha, \beta, \delta) = f(\beta, \gamma, \delta)$

Type 4 for some $\alpha < \beta < \gamma < \delta$, $f(\alpha, \gamma, \delta) = f(\beta, \gamma, \delta)$.

Remark 5.3. By repeatedly applying the Ramsey theorem on ω to eliminate bad tuples of the 4 types above, one can show $\omega_1 \to^{poly} (\omega + k)^3_{l-bdd}$ for any $k, l \in \omega$. This is already in contrast with the dual statements in Ramsey theory.

Question 5.4. *Can we prove in ZFC that $\omega_1 \to^{poly} (\alpha)^3_{2-bdd}$ for any $\alpha < \omega_1$?*

Question 5.5. *Is $\omega_1 \to^{poly} (\omega_1)^3_{2-bdd}$ consistent? Is it a consequence of* PFA*?*

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