MONOCHROMATIC SUMSET WITHOUT THE USE OF LARGE CARDINALS

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Abstract. We show in this note that in the forcing extension by $Add(\omega, \beth_\omega)$, the following Ramsey property holds: for any $r \in \omega$ and any $f : \mathbb{R} \to r$, there exists an infinite $X \subset \mathbb{R}$ such that $X + X$ is monochromatic under f. We also show the Ramsey statement above is true in ZFC when $r = 2$. This answers two questions from [8].

1. Introduction

Definition 1.1. Let $(A, +)$ be an additive structure and κ, r be cardinals. Let $A \to^+ (\kappa)_r$ abbreviate the statement: for any $f : A \to r$, there exists $X \subset A$ with $|X| = \kappa$ such that $X + X =_{def} \{a + b : a, b \in X\}$ is monochromatic under f.

There have been recent developments on additive partition relations for real numbers. Hindman, Leader and Strauss [5] showed that if $2^\omega < \aleph_\omega$ then there exists some $r \in \omega$ such that $\mathbb{R} \not\to^+ (\aleph_0)_r$. On the other hand, Komjáth, Leader, Russell, Shelah, Soukup and Vidnyánszky [8] showed that relative to the existence of an ω_1-Erdős cardinal, it is consistent that for any $r \in \omega$, $\mathbb{R} \to^+ (\aleph_0)_r$. These results are optimal in a sense as there exist the following restrictions:

(1) Komjáth [7] and independently Soukup and Weiss [11] showed that $\mathbb{R} \not\to^+ (\aleph_1)_2$;
(2) Soukup and Vidnyánszky showed there exists a finite coloring of f on $\mathbb{R}$ such that no infinite $X \subset \mathbb{R}$ satisfies that $\underbrace{X + \cdots + X}_{k}$ is monochromatic for $k \geq 3$.

It should be emphasized that the difficulty comes from the fact that repetitions are allowed. If we only want some infinite $X \subset \mathbb{R}$ such that $X \oplus X = \{a + b : a \neq b \in X\}$ is monochromatic, then the classical Ramsey theorem implies this already. In fact, Hindman's finite-sum theorem is a much stronger Ramsey-type statement: any finite coloring of $\mathbb{N}$, there exists some infinite $X \subset \mathbb{N}$ such that $FS(X) =_{def} \{\Sigma_{0 \leq i < k} a_i : \{a_0, a_1, \cdots, a_{k-1}\} \in [X]^{<\omega}\}$ is monochromatic. However, if repeated sums are allowed, things turn towards the other direction: Hindman [4] showed $\mathbb{N} \not\to^+ (\aleph_0)_3$ and Owings asked (and it is still open) that if $\mathbb{N} \not\to^+ (\aleph_0)_2$ is true. Interestingly, Fernández-Bretón and Rinot [3] showed that the uncountable analogs of Hindman's theorem must necessarily fail in a strong way.

2010 *Mathematics Subject Classification.* Primary: 03E02, 03E35, 03E55. I thank Dániel Soukup for telling me about this problem and sharing his insights. I thank the organizers and the participants of the SETTOP 2018 conference in Novi Sad during which part of this work was done. I thank the Department of Mathematical Sciences of Carnegie Mellon University and the US tax payers for its generous support for me to attend SETTOP 2018. I thank James Cummings and Sittinon Jirattikansakul for helpful discussions and comments.

The following questions among others were asked by the authors of [8].

(1) Is the use of large cardinals necessary to establish the consistency $\mathbb{R} \to^+ (\aleph_0)_r$ for all $r \in \omega$?
(2) Is $\mathbb{R} \to^+ (\aleph_0)_2$ true in ZFC?

We answer the first question negatively and the second positively.

Theorem 1.1. *(1) In the forcing extension by $Add(\omega, \beth_\omega)$, $\mathbb{R} \to^+ (\aleph_0)_r$ for any $r \in \omega$.*
(2) $\mathbb{R} \to^+ (\aleph_0)_2$.

Remark 1.2. The continuum in the model of [8] is an $\aleph$-fixed point, which is very large. Over a ground model of GCH, Theorem 1.1 suggests that the most natural way to eliminate the obstacles from cardinal arithmetic works since by a result of Hindman, Leader and Strauss [5], if $\mathbb{R} \to^+ (\aleph_0)_r$ for all $r < \omega$, then $2^\omega \geq \aleph_{\omega+1}$.

Notation 1.3. We will identify $(\mathbb{R}, +)$, as a vector space over $\mathbb{Q}$, with $\bigoplus_{i<2^\omega} \mathbb{Q}$. The latter is the direct sum of 2^ω copies of $(\mathbb{Q}, +)$. More concretely, any $s \in \bigoplus_{i<2^\omega} (\mathbb{Q}, +)$ is a finitely supported function whose range is contained in $\mathbb{Q}$. The addition on the direct sum is defined coordinate-wise. Similarly for some cardinal κ, $\bigoplus_{i<\kappa} \mathbb{N}$ is the direct sum of κ copies of $(\mathbb{N}, +)$. It is easy to see that if $\kappa \leq 2^\omega$, $\bigoplus_{i<\kappa} \mathbb{N}$ is an additive substructure of $\mathbb{R}$.

2. The proof of Theorem 1.1

First we prove part (1). Let $\lambda = \beth_\omega$ and $\mathbb{P} = Add(\omega, \lambda)$. In fact, we show that in $V^{\mathbb{P}}$, $\bigoplus_{i<\lambda} \mathbb{N} \to^+ (\aleph_0)_r$ for any $r \in \omega$.

Definition 2.1. Suppose $W, W' \subset \lambda$ are such that $type(W) = type(W')$. Let $h_{W,W'} : W \to W'$ be the unique order isomorphism. For $A, A' \subset \lambda$ with $type(A) = type(A')$, $h_{A,A'}$ naturally induces a map from $\mathbb{P} \restriction A$ to $\mathbb{P} \restriction A'$ where any $p \in \mathbb{P} \restriction A$ is mapped to $p' \in \mathbb{P} \restriction A'$ such that $dom(p') = h^{-1}_{A,A'}(dom(p))$ and $p'(j) = p(h^{-1}_{A,A'}(j))$. We will abuse the notation by using $h_{A,A'}$ to denote the induced map from $\mathbb{P} \restriction A$ to $\mathbb{P} \restriction A'$. This can be easily inferred from the context.

Definition 2.2 ([6],[5],[8]). For any $r \geq 2$, define a sequence of finite strings of natural numbers $\langle s_l : l \leq r \rangle$ such that for each $l \leq r$, $|s_l| = r + l$ and $s_i(k) = \begin{cases} 2 & \text{if } k < 2l \\ 4 & \text{otherwise.} \end{cases}$ In other words, each s_l is formed by $2l$ many $2's$ followed by $r - l$ many $4's$.

Definition 2.3 (The star operation, see [6],[8]). Let K be either $\mathbb{N}$ or $\mathbb{Q}$. For $k \in \omega$, $s \in (K - \{0\})^k$ and a finite subset of ordinals $a = \{\xi_i : i < k\}_< \subset \lambda$, let $s * a$ denote the function from λ to K supported on a that sends ξ_i to $s(i)$.

We will use the following combinatorial lemma due to Shelah [10], [9].

Lemma 2.4 (The higher dimensional Δ-system lemma). *Fix $r, d \in \omega$. Let $\langle \dot{d}_i : [\lambda]^i \to r | i \leq d \rangle$ be a sequence of $\mathbb{P}$-names for colorings. Then there exists $E \subset \lambda$ of order type ω_1 and $W : [E]^{\leq d} \to [\lambda]^{\leq \aleph_0}$ such that*

CL.1 *For all $u \in [E]^{\leq d}$, $u \subset W(u)$ and $\mathbb{P} \restriction W(u)$ contains a maximal antichain deciding the value of $\dot{d}_{|u|}(u)$.*

CL.2 *For any $u, v \in [E]^{\leq d}$ such that $|u| = |v|$, type$(W(u)) =$ type$(W(v))$, $h_{W(u),W(v)}(u) = v$ and for any $p \in \mathbb{P} \restriction W(u)$, for any $n < r$, $p \Vdash \dot{d}_{|u|}(u) = n \Leftrightarrow h_{W(u),W(v)}(p) \Vdash \dot{d}_{|v|}(v) = n$.*

CL.3 *For any $u, v \in [E]^{\leq d}$, $W(u) \cap W(v) = W(u \cap v)$.*

CL.4 *For any $u_1 \subset u_2, u_1' \subset u_2'$ where $u_2, u_2' \in [E]^{\leq d}$, if $(u_2, u_1, <) \simeq (u_2', u_1', <)$, then $h_{W(u_2),W(u_2')} \restriction W(u_1) = h_{W(u_1),W(u_1')}$.*

Remark 2.5. Different versions of Lemma 2.4 appeared in [10], Lemma 4.1 of [9], Claim 7.2.a of [1] and the appendix of [12]. We will use the fact that $\lambda = \beth_\omega$ to present a slightly simpler proof, following the arguments presented in Claim 7.2.a of [1]. More specifically, we will take advantage of the following fact: there exists λ_0 such that $\lambda \to (\lambda_0)^{2d}_{2^\omega}$ and $\lambda_0 \to (\aleph_1)^{2d}_{2^\omega}$. This statement is the only fact about λ we will use in the proof. In fact, that $\lambda \to (\aleph_1)^{2d}_{2^\omega}$ suffices to get the conclusion of Lemma 2.4 but the proof is slightly more complicated. The interested readers are directed to the proofs in Claim 7.2.a of [1] (for **CL.1**,**CL.2**,**CL.3**) and in the appendix of [12] (for **CL.4**).

Proof. Fix $r, d \in \omega$ and $\langle \dot{d}_i : i \leq d+1 \rangle$ as in Lemma 2.4 and λ_0 as in Remark 2.5. Call a function f *monotone* if whenever $u \subset v \in dom(f)$, we have $f(u) \subset f(v)$.

Claim 2.6. *For any $E^* \subset \lambda$ of order type $\kappa \leq \lambda$, $\kappa_0 \geq \omega_1$ such that $\kappa \to (\kappa_0)^{2d}_{2^\omega}$ and any monotone $W' : [E^*]^{\leq d} \to [\lambda]^{\leq \aleph_0}$ such that for all $u \in [E^*]^{\leq d}$, $u \subset W'(u)$ and $\mathbb{P} \restriction W'(u)$ contains a maximal antichain deciding the value of $\dot{d}_{|u|}(u)$, there exists $E' \subset E^*$ of order type κ_0 such that **CL.1**, **CL.2** hold with E, W replaced by E', W' and the following holds: for any $k \in \omega$ and any $\{u_i \in [E']^{\leq d} : i < k\}$ and $\{v_i \in [E']^{\leq d} : i < k\}$, if*

$$(\bigcup_{i<k} u_i, u_0, \cdots, u_{k-1}, <) \simeq (\bigcup_{i<k} v_i, v_0, \cdots, v_{k-1}, <),$$

then the isomorphism can be extended to one that witnesses

$$(\bigcup_{i<k} W'(u_i), W'(u_0), \cdots, W'(u_{k-1}), <) \simeq (\bigcup_{i<k} W'(v_i), W'(v_0), \cdots, W'(v_{k-1}), <).$$

*In particular, **CL.4** holds with E, W replaced by E', W'.*

Proof of the claim. Define an equivalence relation $\sim$ on $[E^*]^{2d}$ as follows: $u \sim v$ iff

(1) whenever $u' \in [u]^{\leq d}$ and $v' \in [v]^{\leq d}$ are such that $(u, u', <) \simeq (v, v', <)$ (which in particular implies there is some $k \leq d$, $|u'| = |v'| =_{def} k$), we have that $(W'(u'), u', <) \simeq (W'(v'), v', <)$ and for any $p \in \mathbb{P} \restriction W'(u')$ and $n < r$, $p \Vdash \dot{d}_k(u') = n$ iff $h_{W'(u'),W'(v')}(p) \Vdash \dot{d}_k(v') = n$.

(2) whenever $u_0, u_1 \in [u]^{\leq d}$ and $v_0, v_1 \in [v]^{\leq d}$ satisfy that $(u, u_0, u_1, <) \simeq (v, v_0, v_1, <)$, then

$$(\bigcup_{a \in [u]^{\leq d}} W'(a), W'(u_0), W'(u_1), u, u_0, u_1) \simeq (\bigcup_{b \in [v]^{\leq d}} W'(b), W'(v_0), W'(v_1), v, v_0, v_1).$$

It can be easily checked that the number of equivalence classes is at most 2^ω. By the fact that $\kappa \to (\kappa_0)^{2d}_{2^\omega}$, we can find $E' \subset E^*$ of order type κ_0 such that elements in $[E']^{2d}$ are mutually $\sim$-equivalent. It is clear that **CL.1** and **CL.2** hold. Fix $k \in \omega$ and $\{u_i \in [E']^{\leq d} : i < k\}$ and $\{v_i \in [E']^{\leq d} : i < k\}$ such that

$$(\bigcup_{i<k} u_i, u_0, \cdots, u_{k-1}, <) \simeq (\bigcup_{i<k} v_i, v_0, \cdots, v_{k-1}, <).$$

(2) in the definition of $\sim$ ensures that for $i < j < k$, by the fact that $(u_i \cup u_j, u_i, u_j, <) \simeq (v_i \cup v_j, v_i, v_j, <)$, we have

$$(W'(u_i) \cup W'(u_j), W'(u_i), W'(u_j), u_i, u_j, <) \simeq (W'(v_i) \cup W'(v_j), W'(v_i), W'(v_j), v_i, v_j, <).$$

Therefore, it is easy to see $\bigcup_{i<k} h_{W'(u_i),W'(v_i)}$, extending the unique isomorphism from $(\bigcup_{i<k} u_i, u_0, \cdots, u_{k-1}, <)$ to $(\bigcup_{i<k} v_i, v_0, \cdots, v_{k-1}, <)$, is an isomorphism between

$$(\bigcup_{i<k} W'(u_i), W'(u_0), \cdots, W'(u_{k-1}), <)$$

and

$$(\bigcup_{i<k} W'(v_i), W'(v_0), \cdots, W'(v_{k-1}), <).$$

$\square$

Let $W_0 : [\lambda]^{\leq d} \to [\lambda]^{\leq \aleph_0}$ be a monotone function such that for all $u \in [\lambda]^{\leq d}$, $u \subset W_0(u)$ and $\mathbb{P} \restriction W_0(u)$ contains a maximal antichain deciding the value of $\dot{d}_{|u|}(u)$. This is possible by the c.c.c-ness of $\mathbb{P}$. Apply Claim 2.6 with $E^* = \lambda, \kappa = \lambda, \kappa_0 = \lambda_0$ and $W' = W_0$ to get $E_0 \subset \lambda$ of order type λ_0.

For each $u \in [E_0]^{\leq d}$, define $W(u) =_{def} \bigcup\{\bigcap_{v \in X} W_0(v) : X \subset [E_0]^{\leq d}, \bigcap X \subset u\}$. Notice that for any $u \in [E_0]^{\leq d}$, $W_0(u) \subset W(u)$.

Claim 2.7. *(1) For any $u, v \in [E_0]^{\leq d}$, $W(u) \cap W(v) = W(u \cap v)$ so in particular W is monotone and*
(2) for any $u \in [E_0]^{\leq d}$, $W(u)$ is a countable subset of λ.

Proof of the claim. (1) immediately follows from the definition. To see (2) holds, fix $u \in [E_0]^{\leq d}$. First notice that in the definition of $W(u)$, it suffices to consider those $X \subset [E_0]^{\leq d}$ such that $|X| \leq d+1$. To see this, it suffices to note that for any $X \subset [E_0]^{\leq d}$ with $\bigcap X \subset u$, there exists $Y \subset X$ such that $|Y| \leq d+1, \bigcap Y = \bigcap X$ (in particular, $\bigcap_{v \in X} W_0(v) \subset \bigcap_{u \in Y} W_0(u)$). If $|X| \leq d$, take $Y = X$. Otherwise, pick some $x \in X$, then $x \in [E_0]^{\leq d}$. For each $\xi \in x$, if $\xi \notin \bigcap X$, then there exists $x_\xi \in X$ such that $\xi \notin x_\xi$. Let $Y = \{x\} \cup \{x_\xi : \xi \in x - \bigcap X\}$. This Y as defined clearly satisfies the requirement.

The following suffices for the claim: for any $k \leq d$ and any $X =_{def} \{x_0, \cdots, x_k\}$ $X' =_{def} \{x'_0, \cdots, x'_k\} \subset [E_0]^{\leq d}$ with $\bigcap X, \bigcap X' \subset u$, if $u \cap \bigcup X = u \cap \bigcup X'$ and

$$(\bigcup X, u \cap \bigcup X, x_0, \cdots, x_k, <) \simeq (\bigcup X', u \cap \bigcup X', x'_0, \cdots, x'_k, <) \tag{2.8}$$

then $\bigcap_{x \in X} W_0(x) = \bigcap_{x' \in X'} W_0(x')$. If the assertion is true, $W(u)$ will be a finite union of countable sets. To see this, each structure $(\bigcup X, u \cap \bigcup X, x_0, \cdots, x_k, <)$ is uniquely coded by a finite function from $|\bigcup X|$ to 2^{k+2}. Clearly the number of such codes is finite. Structures of the same code are isomorphic in the sense of (2.8).

To prove the assertion, fix X, X' as above and let $\bar{u} = u \cap \bigcup X = u \cap \bigcup X'$. If $\bigcup X = \bigcup X'$, then by (2.8), $X = X'$, we are done. So we may assume $\bigcup X \neq \bigcup X'$. We will induct on the size of $(\bigcup X) \Delta (\bigcup X')$. Let $\xi \in \bigcup X, \xi' \in \bigcup X'$ be such that $(\bigcup X) \cap \xi = (\bigcup X') \cap \xi'$ but $\xi \notin \bigcup X'$ or $\xi' \notin \bigcup X$. We may without loss of generality assume $\xi < \xi'$. In this case, $\xi \notin \bigcup X'$. In particular, $\xi \notin \bar{u}$ and by (2.8), $\xi' \notin \bar{u}$. Let $X'' = \{x''_i : i \leq k\}$ such that $x''_i = \begin{cases} x'_i & \xi' \notin x'_i \\ (x'_i - \{\xi'\}) \cup \{\xi\} & \xi' \in x'_i. \end{cases}$ It is clear that

$$(\bigcup X'', u \cap \bigcup X'', x''_0, \cdots, x''_k, <) \simeq (\bigcup X', u \cap \bigcup X', x'_0, \cdots, x'_k, <). \tag{2.9}$$

It suffices to show $\bigcap_{x'' \in X''} W_0(x'') = \bigcap_{x' \in X'} W_0(x')$ since $|(\bigcup X)\Delta(\bigcup X'')| < |(\bigcup X)\Delta(\bigcup X')|$ so we can finish by the induction hypothesis. There exists $j \leq k$ such that $\xi' \notin x'_j$ since otherwise $\xi' \in \bigcap X' \subset u \cap \bigcup X' = \bar{u}$ which cannot be true. Thus $x''_j = x'_j$. By Claim 2.6, there exists an isomorphism h from $(\bigcup_{i \leq k} W_0(x'_i), W_0(x'_0), \cdots, W_0(x'_k), <)$ to $(\bigcup_{i \leq k} W_0(x''_i), W_0(x''_0), \cdots, W_0(x''_k), <)$ extending the unique isomorphism:

$$(\bigcup_{i \leq k} x'_i, x'_0, \cdots, x'_k, <) \simeq (\bigcup_{i \leq k} x''_i, x''_0, \cdots, x''_k, <).$$

Since $x''_j = x'_j$ and h sends $W_0(x'_j)$ onto $W_0(x''_j)$, we know $h \restriction W_0(x'_j)$ is the identity function on $W_0(x'_j)$. Therefore, $W_0(x'_j) \supset \bigcap_{x' \in X'} W_0(x') = h(\bigcap_{x' \in X'} W_0(x')) = \bigcap_{x'' \in X''} W_0(x'')$.

□

Finally, using $\lambda_0 \to (\aleph_1)^{2d}_{2^\omega}$ we apply Claim 2.6 with $E^* = E_0, \kappa = \lambda_0, \kappa_0 = \omega_1$ and $W' = W$ to get $E \subset E_0$ of order type ω_1 such that **CL.1**, **CL.2**, **CL.4** hold for E and W. **CL.3** also holds by Claim 2.7.

□

Let $G \subset \mathbb{P}$ be generic over V. In $V[G]$, suppose $f : \bigoplus_{i<\lambda} \mathbb{N} \to r$ is the given coloring. Define $d_i : [\lambda]^{r+i} \to r$ such that $d_i(\bar{a}) = f(s_i * \bar{a})$ for $i \leq r$. Let $\dot{d}_i$ for $i \leq r$ be the corresponding names.

Back in V, apply Lemma 2.4 to $d = 2r$ and $\langle \dot{d}_i : i \leq r \rangle$, and find the desired E and W (strictly speaking, we should apply to the sequence $\langle \dot{d}'_{i+r} : i \leq r \rangle$ where $\dot{d}'_{i+r} = \dot{d}_i$ for $i \leq r$). Enumerate E increasingly as $\{e_i : i \in \omega_1\}$. Let $A_i = \{e_{\omega \cdot i + j} : 1 \leq j \leq \omega\}$ for each $i < r$. For each $i < r, j \leq \omega$, let $\alpha^i_j = e_{\omega \cdot i + (1+j)}$.

Definition 2.10. For any $l \leq r$ and any tuple $\bar{s} \in \Pi_{i<l}[A_i]^2 \times \Pi_{i \geq l, i<r} A_i$, we naturally identify $\bar{s}$ as an $(r+l)$-tuple. To be more concrete, we take 2 elements from each of the first l sets ordered naturally and 1 element from each of the remaining sets.

(1) $\bar{s}$ is *l-canonical* if $\bar{s}$ is of the form

$$(\alpha^0_{i_0}, \alpha^0_{i'_0}, \cdots, \alpha^{l-1}_{i_{l-1}}, \alpha^{l-1}_{i'_{l-1}}, \alpha^l_{i_l}, \alpha^{l+1}_{i_{l+1}}, \cdots, \alpha^{r-1}_{i_{r-1}})$$

such that for any $k < l$, $i_k < i'_k \leq \omega$ and $\max\{i_m : m < r, i_m < \omega\} < i'_k$ for any $k < l$. If, in addition, we are given a sequence $\langle D_i \subset A_i : i < r \rangle$, then we say $\bar{s}$ is *from* $\langle D_i : i < r \rangle$ if $\bar{s} \in \Pi_{i<l}[D_i]^2 \times \Pi_{i \geq l, i<r} D_i$.

(2) We call $\bar{i} = \langle i_k : k < r \rangle$ the *index* of an l-canonical tuple $\bar{s}$. $\bar{s}$ is *index-strictly-increasing* if whenever $k < k' < r$, $i_k \leq i_{k'}$ and if $i_k \in \omega$, then $i_k < i_{k'}$.

(3) For any two ordinals α, α', let $\bar{s}_{\alpha \to \alpha'}$ denote the tuple obtained by replacing the occurrence of α in $\bar{s}$ by α'. Similarly for any two sequences of ordinals

$\bar{\alpha}, \bar{\alpha}'$ of the same length, $\bar{s}_{\bar{\alpha}\to\bar{\alpha}'}$ denotes the tuple obtained by replacing the occurence of α_i in $\bar{s}$ by α'_i for each $i < |\bar{\alpha}|$.

Notation 2.11. Many times in what follows, we confuse a tuple with the set that consists of elements from the tuple, namely $\bar{s} = \langle s_i : i < n\rangle$ is identified with $\{s_i : i < n\}$. It can be mostly inferred from the context, for example $W(\bar{s}) = W(\{s_i : i < n\})$ and $W(\bar{s} \cap \bar{t}) = W(\{s_i : i < n\} \cap \{t_j : j < m\})$ where $\bar{t} = \langle t_j : j < m\rangle$.

Claim 2.12. *In $V[G]$, for any $j < r$ and for any finite $B_i \subset A_i$ with $a^i_\omega \in B_i$ for $i < r$, there exists arbitrarily large $\alpha \in A_j \backslash \{\alpha^j_\omega\}$ such that $\alpha > B_j \backslash \{\alpha^j_\omega\}$ and the following is true: for any $l \leq r$, any l-canonical tuple $\bar{s}$ from $\langle B_i : i < r\rangle$ containing α^j_ω, $d_l(\bar{s}') = d_l(\bar{s})$ where $\bar{s}' = \bar{s}_{\alpha^j_\omega \to \alpha}$.*

Proof. Fix $j < r$. Work in V. For any given $p \in \mathbb{P}$ and $\gamma \in A_j \backslash \{\alpha^j_\omega\}$, we want to find $p' \leq p$ and $\alpha > \max\{\gamma, \max B_j \backslash \{\alpha^j_\omega\}\}$ in $A_j \backslash \{\alpha^j_\omega\}$ such that p' forces the conclusion above is true for this α. This clearly suffices by the density argument.

Given $p \in \mathbb{P}$, extending it if necessary, we may assume that for each $l \leq r$ and each l-canonical tuple $\bar{s}$ from $\langle B_i : i < r\rangle$, $p \restriction W(\bar{s})$ decides the value of $\dot{d}_l(\bar{s})$. Find $\alpha \in A_j \backslash \{\alpha^j_\omega\}$ large enough such that

- $\alpha > \max\{\max B_j \backslash \{\alpha^j_\omega\}, \gamma\}$
- $dom(p) \cap (W(u \cup \{\alpha\}) - W(u)) = \emptyset$ for all $u \in [\bigcup_{i<r} B_i]^{\leq 2r-1}$.

This is possible since $dom(p)$ is finite and for any fix $u \in [\bigcup_{i<r} B_i]^{\leq 2r-1}$, $W(u \cup \{\alpha\}) \cap W(u \cup \{\alpha'\}) = W(u)$ for any $\alpha \neq \alpha' > \max u + 1$.

Define $p' = p \cup \bigcup_{l \leq r}\{h_{W(\bar{s}),W(\bar{s}')}(p \restriction W(\bar{s})) : \bar{s}$ is an l-canonical tuple from $\langle B_i : i < r\rangle, \alpha^j_\omega \in \bar{s}$, and $\bar{s}' = \bar{s}_{\alpha^j_\omega \to \alpha}\}$. We claim that p' is the desired condition. To verify this, it suffices to show the following:

(1) p' is a condition. We do this by showing for p is compatible with $h_{W(\bar{s}),W(\bar{s}')}(p \restriction W(\bar{s}))$ and $h_{W(\bar{s}),W(\bar{s}')}(p \restriction W(\bar{s}))$ is compatible with $h_{W(\bar{t}),W(\bar{t}')}(p \restriction W(\bar{t}))$ for each $\bar{s}, \bar{t}$ as above.
   - Fix $\bar{s}$. To see p is compatible with $p^* =_{def} h_{W(\bar{s}),W(\bar{s}')}(p \restriction W(\bar{s}))$, notice that $dom(p) \cap dom(p^*) \subset dom(p) \cap W(\bar{s}') \subset W(\bar{s}' - \{\alpha\})$ by the choice of α. By **CL.4**, $h_{W(\bar{s}),W(\bar{s}')} \restriction W(\bar{s} - \{\alpha^j_\omega\})$ is the identity function on $W(\bar{s} - \{\alpha^j_\omega\})$ since $(\bar{s}, \bar{s} - \{\alpha^j_\omega\}, <) \simeq (\bar{s}', \bar{s}' - \{\alpha\}, <)$ and $\bar{s} - \{\alpha^j_\omega\} = \bar{s}' - \{\alpha\}$. Hence $p^* \restriction W(\bar{s}' - \{\alpha\}) = p \restriction W(\bar{s} - \{\alpha^j_\omega\}) = p \restriction W(\bar{s}' - \{\alpha\})$.
   - Fix $\bar{s}, \bar{t}$ as above. Let $q_0 =_{def} h_{W(\bar{s}),W(\bar{s}')}(p \restriction W(\bar{s})), q_1 =_{def} h_{W(\bar{t}),W(\bar{t}')}(p \restriction W(\bar{t}))$. Notice that $dom(q_0) \cap dom(q_1) \subset W(\bar{s}') \cap W(\bar{t}') = W(\bar{s}' \cap \bar{t}') = W((\bar{s} \cap \bar{t})_{\alpha^j_\omega \to \alpha})$. Observe that $(\bar{s}, \bar{s} \cap \bar{t}, <) \simeq (\bar{s}', \bar{s}' \cap \bar{t}', <)$ and $(\bar{t}, \bar{s} \cap \bar{t}, <) \simeq (\bar{t}', \bar{s}' \cap \bar{t}', <)$. By **CL.4**, we have $h_{W(\bar{s}),W(\bar{s}')}(W(\bar{s} \cap \bar{t})) = W(\bar{s}' \cap \bar{t}')$ and $h_{W(\bar{t}),W(\bar{t}')}(W(\bar{s} \cap \bar{t})) = W(\bar{s}' \cap \bar{t}')$. Hence $q_0 \restriction W((\bar{s} \cap \bar{t})_{\alpha^j_\omega \to \alpha}) = h_{W(\bar{s}),W(\bar{s}')}(p \restriction W(\bar{s} \cap \bar{t})) = h_{W(\bar{s} \cap \bar{t}),W(\bar{s}' \cap \bar{t}')}(p \restriction W(\bar{s} \cap \bar{t}))$ and $q_1 \restriction W((\bar{s} \cap \bar{t})_{\alpha^j_\omega \to \alpha}) = h_{W(\bar{t}),W(\bar{t}')}(p \restriction W(\bar{s} \cap \bar{t})) = h_{W(\bar{s} \cap \bar{t}),W(\bar{s}' \cap \bar{t}')}(p \restriction W(\bar{s} \cap \bar{t}))$. Since q_0 and q_1 agree on their common domain, it follows that they are compatible.

(2) p' forces $\dot{d}_l(\bar{s}) = \dot{d}_l(\bar{s}')$ for any l-canonical tuple $\bar{s}$ from $\langle B_i : i < r\rangle$ containing α^j_ω where $\bar{s}' = \bar{s}_{\alpha^j_\omega \to \alpha}$ for any $l \leq r$. Fix l and $\bar{s}$. By the initial assumption about p, we know there exists $n < r$ such that $p \restriction W(\bar{s}) \Vdash \dot{d}_l(\bar{s}) = n$. By **CL.2**, $h_{W(\bar{s}),W(\bar{s}')}(p \restriction W(\bar{s})) \Vdash \dot{d}_l(\bar{s}') = n$. Hence $p' \Vdash \dot{d}_l(\bar{s}') = n = \dot{d}_l(\bar{s})$.

□

Claim 2.13. *There exist $C^i \subset A_i$ containing α^i_ω for $i < r$ such that*

(1) for each $i < r$, $type(C^i) = \omega + 1$

(2) for each $l \leq r$ and each index-strictly-increasing l-canonical tuple

$$\bar{s} = (\alpha^0_{i_0}, \alpha^0_{i'_0}, \cdots, \alpha^{l-1}_{i_{l-1}}, \alpha^{l-1}_{i'_{l-1}}, \alpha^l_{i_l}, \alpha^{l+1}_{i_{l+1}}, \cdots, \alpha^{r-1}_{i_{r-1}})$$

from $\langle C^i : i < r\rangle$, $d_l(\bar{s}) = d_l(\bar{s}')$, where

$$\bar{s}' = (\alpha^0_{i_0}, \alpha^0_\omega, \cdots, \alpha^{l-1}_{i_{l-1}}, \alpha^{l-1}_\omega, \alpha^l_\omega, \alpha^{l+1}_\omega, \cdots, \alpha^{r-1}_\omega).$$

In particular, the color $\bar{s}$ gets under d_l only depends on its index.

Proof. We will build these sets in ω-steps. We will pick one point at a time from sets listed in the following order:

$$A_0, A_1, \cdots, A_{r-1}, A_0, A_1, \cdots, A_{r-1}, A_0, A_1, \cdots, A_{r-1}, \cdots.$$

In particular, we will find $J^i = \{j^i_k : k \in \omega\} \subset \omega$ such that $C^i = \{\alpha^i_{j^i_k} : k \in \omega\} \cup \{\alpha^i_\omega\}$ for each $i < r$. For fixed i, k, let C^i_k denote $\{\alpha^i_{j^i_{k'}} : k' < k\} \cup \{\alpha^i_\omega\}$.

Recall $C^i_0 = \{\alpha^i_\omega\}$ for all $i < r$. Recursively, suppose for some $i < r$ and $k \in \omega$ we have defined C^p_q for all $\langle q, p\rangle <_{lex} \langle k, i\rangle$ (i.e. either $q < k$ or $q = k$ and $p < i$). Apply Claim 2.12 to pick $j^i_k \in \omega$ such that

- $j^i_k > j^p_q$ for all $\langle q, p\rangle <_{lex} \langle k, i\rangle$
- for any $l \leq r$ and any l-canonical tuple $\bar{s}$ containing α^i_ω from $\langle C^p_{k_p} : p < r\rangle$ where $k_p = k$ if $p < i$ and $k_p = k - 1$ if $p \geq i$, it is true that $d_l(\bar{s}) = d_l(\bar{s}_{\alpha^i_\omega \to \alpha^i_{j^i_k}})$.

We now verify that $\langle C^i : i < r\rangle$ satisfies (2). Fix $l \leq r$ and some index-strictly-increasing l-canonical tuple $\bar{s}$ from $\langle C^i : i < r\rangle$, say

$$\bar{s} = (\alpha^0_{i_0}, \alpha^0_{i'_0}, \cdots, \alpha^{l-1}_{i_{l-1}}, \alpha^{l-1}_{i'_{l-1}}, \alpha^l_{i_l}, \alpha^{l+1}_{i_{l+1}}, \cdots, \alpha^{r-1}_{i_{r-1}}).$$

By the hypothesis, we know $\max\{i_m : m < r, i_m < \omega\} < i'_k$ for any $k < l$. By the conclusion of Claim 2.12 and the index management in our recursive process, we know that

$$d_l(\bar{s}) = d_l(\alpha^0_{i_0}, \alpha^0_\omega, \cdots, \alpha^{l-1}_{i_{l-1}}, \alpha^{l-1}_\omega, \alpha^l_\omega, \alpha^{l+1}_\omega, \cdots, \alpha^{r-1}_\omega).$$

□

By Claim 2.13, we may without loss of generality assume that the sets $\langle A_i : i < r\rangle$ already satisfy that: for each $l \leq r$, for each index-strictly-increasing l-canonical tuple $\bar{s}$ from $\langle A_i : i < r\rangle$ satisfies (2) in the conclusion of Claim 2.13.

To finish the proof, we basically need similar arguments as in Claim 2.9 and Step 5 from [8]. We supply a proof for completeness.

Claim 2.14. *There exist $\langle B_i \subset A_i : i < r, \alpha^i_\omega \in B_i, type(B_i) = \omega + 1\rangle$ and $\langle \rho_l < r : l \leq r\rangle$ such that for each $l \leq r$, for each index-strictly-increasing l-canonical tuple $\bar{s}$ from $\langle B_i : i < r\rangle$, $d_l(\bar{s}) = \rho_l$.*

Proof. Fix $l \leq r, W \in [\omega]^{\aleph_0}$. Define $g : [W]^l \to r$ such that for each $\bar{i} = \{i_0 < i_1 < \cdots < i_{l-1}\}$,

$$g(\bar{i}) = d_l(\alpha^0_{i_0}, \alpha^0_\omega, \cdots, \alpha^{l-1}_{i_{l-1}}, \alpha^{l-1}_\omega, \alpha^l_\omega, \alpha^{l+1}_\omega, \cdots, \alpha^{r-1}_\omega).$$

Let $I \in [W]^{\aleph_0}$ be a monochromatic subset with color ρ_l for g. For any index-strictly-increasing l-canonical tuple

$$\bar{s} = (\alpha^0_{j_0}, \alpha^0_{j'_0}, \cdots, \alpha^{l-1}_{j_{l-1}}, \alpha^{l-1}_{j'_{l-1}}, \alpha^l_{j_l}, \alpha^{l+1}_{j_{l+1}}, \cdots, \alpha^{r-1}_{j_{r-1}})$$

such that $j_k, j'_t \in I \cup \{\omega\}$ for any $k < r$ and $t < l$, by Claim 2.13 and the remark that follows, we know that

$$d_l(\bar{s}) = d_l(\alpha^0_{j_0}, \alpha^0_\omega, \cdots, \alpha^{l-1}_{j_{l-1}}, \alpha^{l-1}_\omega, \alpha^l_\omega, \alpha^{l+1}_\omega, \cdots, \alpha^{r-1}_\omega) = g(\{j_0 < \cdots < j_{l-1}\}) = \rho_l.$$

To get the conclusion of the claim, apply the procedure above repeatedly to get $\omega \supset I_0 \supset I_1 \supset \cdots \supset I_{r-1} \supset I_r$. It is clear that $B_i = \{\alpha^i_j : j \in I_r\} \cup \{\alpha^i_\omega\}$ for $i < r$ will be the desired sets.

□

By Claim 2.14, we may without loss of generality assume that the sets $\langle A_i : i < r\rangle$ already satisfy that: there exist $\langle \rho_l : l \le r\rangle$ such that for each $l \le r$, for each index-strictly-increasing l-canonical tuple $\bar{s}$ from $\langle A_i : i < r\rangle$, $d_l(\bar{s}) = \rho_l$. By the Pigeon hole principle, there exist $l' < l$ such that $\rho_{l'} = \rho_l = \rho$.

Claim 2.15. *There exists an infinite X such that $f \restriction X + X \equiv \rho$.*

Proof. For $i < \omega$, let

$$\begin{aligned} \bar{a}_i = (\alpha^0_0, \alpha^0_\omega, \alpha^1_1, \alpha^1_\omega, \cdots, \alpha^{l'-1}_{l'-1}, \alpha^{l'-1}_\omega, \\ \alpha^{l'}_{l'+i\cdot(l-l')}, \alpha^{l'+1}_{l'+1+i\cdot(l-l')}, \cdots, \alpha^{l-1}_{l-1+i\cdot(l-l')}, \\ \alpha^l_\omega, \cdots, \alpha^{r-1}_\omega), \end{aligned}$$

namely, we take

(1) $\{\alpha^k_k, \alpha^k_\omega\}$ from A_k for each $k < l'$
(2) $\{\alpha^k_{k+i(l-l')}\}$ from A_k for each $k \ge l'$ and $k < l$
(3) $\{\alpha^k_\omega\}$ from A_k for each $k \ge l$.

Define $x_i = \frac{1}{2} s_{l'} * \bar{a}_i$. For $i < j \in \omega$, consider

$$\begin{aligned} \bar{b}_{i,j} = (\alpha^0_0, \alpha^0_\omega, \cdots, \alpha^{l'-1}_{l'-1}, \alpha^{l'-1}_\omega, \\ \alpha^{l'}_{l'+i\cdot(l-l')}, \alpha^{l'}_{l'+j\cdot(l-l')}, \cdots \alpha^{l-1}_{l-1+i\cdot(l-l')}, \alpha^{l-1}_{l-1+j\cdot(l-l')}, \\ \alpha^l_\omega, \cdots, \alpha^{r-1}_\omega), \end{aligned}$$

namely, we take

(1) $\{\alpha^k_k, \alpha^k_\omega\}$ from A_k for each $k < l'$
(2) $\{\alpha^k_{k+i(l-l')}, \alpha^k_{k+j(l-l')}\}$ from A_k for each $k \ge l'$ and $k < l$
(3) $\{\alpha^k_\omega\}$ from A_k for each $k \ge l$.

It is not hard to notice that $x_i + x_j = s_l * \bar{b}_{i,j}$.

For any $i < j \in \omega$, $\bar{a}_i$ ($\bar{b}_{i,j}$ respectively) is easily seen to be an index-strictly-increasing l'-canonical (l-canonical) tuple. Therefore, $f(2x_i) = f(s_{l'} * \bar{a}_i) = d_{l'}(\bar{a}_i) = \rho_{l'} = \rho$ and $f(x_i + x_j) = f(s_l * \bar{b}_{i,j}) = d_l(\bar{b}_{i,j}) = \rho_l = \rho$. We conclude that $X = \{x_i : i \in \omega\}$ is the set as desired.

□

Claim 2.15 finishes the proof of (1).

Proof of part (2). We prove a stronger statement: $\bigoplus_{i<\omega_1} \mathbb{N} \to^+ (\aleph_0)_2$. To see this, for any such f, let $d_i(\bar{a}) = f(s_i * \bar{a})$ be defined as before for $i < 3$. In particular, the domain of d_i is $[\omega_1]^{i+2}$ for $i < 3$. Apply the Dushnik-Miller theorem (see Theorem 11.3 in [2]) to get $A = \{\alpha_j : j \leq \omega\} \in [\omega_1]^{\omega+1}$ such that $d_i \restriction [A]^{i+2} \equiv \rho_i < 2$ for all $i < 3$. By the Pigeon hole principle we have the following cases and we will define $X = \{x_i : i \in \omega\}$ for each case.

(1) $\rho_0 = \rho_1 = \rho$. Let $x_i = \frac{1}{2} s_0 * (\alpha_i, \alpha_\omega)$. Then $f(2x_i) = f(s_0 * (\alpha_i, \alpha_\omega)) = d_0(\alpha_i, \alpha_\omega) = \rho_0 = \rho$. For any $i < j \in \omega$, $f(x_i + x_j) = f(s_1 * (\alpha_i, \alpha_j, \alpha_\omega)) = d_1(\alpha_i, \alpha_j, \alpha_\omega) = \rho_1 = \rho$.

(2) $\rho_0 = \rho_2 = \rho$. Let $x_i = \frac{1}{2} s_0 * (\alpha_{2i}, \alpha_{2i+1})$. Then $f(2x_i) = f(s_0 * (\alpha_{2i}, \alpha_{2i+1})) = d_0(\alpha_{2i}, \alpha_{2i+1}) = \rho_0 = \rho$. For any $i < j \in \omega$, $f(x_i + x_j) = f(s_2 * (\alpha_{2i}, \alpha_{2i+1}, \alpha_{2j}, \alpha_{2j+1})) = d_2(\alpha_{2i}, \alpha_{2i+1}, \alpha_{2j}, \alpha_{2j+1}) = \rho_2 = \rho$.

(3) $\rho_2 = \rho_1 = \rho$. Let $x_i = \frac{1}{2} s_0 * (\alpha_0, \alpha_1, \alpha_{i+2})$. Then $f(2x_i) = f(s_0 * (\alpha_0, \alpha_1, \alpha_{i+2})) = d_0(\alpha_0, \alpha_1, \alpha_{i+2}) = \rho_0 = \rho$. For any $i < j \in \omega$, $f(x_i + x_j) = f(s_2 * (\alpha_0, \alpha_1, \alpha_{i+2}, \alpha_{j+2})) = d_2(\alpha_0, \alpha_1, \alpha_{i+2}, \alpha_{j+2}) = \rho_2 = \rho$.

□

Clearly the proof above does not generalize to the case when $r = 3$ since $2^\omega \not\to (\omega+2)^3_2$. A more fundamental restriction is that by a result of Hindman, Leader and Strauss [5], there exists some $r \in \omega$ such that $\bigoplus_{i<\omega_1} \mathbb{N} \not\to^+ (\aleph_0)_r$.

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