Weakly Represented Families in Reverse Mathematics*

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Abstract. We study the proof strength of various second order logic principles that make statements about families of sets and functions. Usually, families of sets or functions are represented in a uniform way by a single object. In order to be able to go beyond the limitations imposed by this approach, we introduce the concept of weakly represented families of sets and functions. This allows us to study various types of families in the context of reverse mathematics that have been studied in set theory before. The results obtained witness that the concept of weakly represented families is a useful and robust tool in reverse mathematics.

1 Introduction

The study of cardinal invariants of the continuum is an important and well-studied branch of set-theory. A cardinal invariant is a cardinal that lies between ω_1 and the continuum $2^{\aleph_0}$. Their study has been important both for forcing theory and for the development of techniques for constructing certain special sets of real numbers in ZFC.

In this work we try to formulate analogues of some of these cardinal invariants in the context of models of second order arithmetic and reverse mathematics. Consider a model of second order arithmetic $(M, S, +, \cdot, 0, 1)$. The basic idea of the present study is that if a suitably "nice" coding of a set of subsets of M satisfying certain combinatorial properties is present in the second order part S

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of this model, then this corresponds to the set-theoretic statement that a certain cardinal invariant of the continuum is small. The notion of "nice coding" that we will use is that of weakly represented families, the definition of which will be made precise in Definition 4.

In the next section we give a short introduction to reverse mathematics, which will then allow us to formulate the second order arithmetical principles that we wish to study in Section 3. In Section 4 we can then discuss the connections with cardinal invariants.

We point out that connections between recursion theory and cardinal invariants have previously been studied by Rupprecht [37] as well as by Brendle, Brooke-Taylor, Ng and Nies [3]; however, their work is only loosely related to the present study.

2 Second order arithmetic and its base system

Second order arithmetic is the two-sorted strengthening of first order logic, that is, it is obtained as follows: We introduce set variables in addition to the number variables existing in first order logic. The function and relation symbols "$\cdot$", "$+$", "$=$" and "$<$" of the language of first order logic remain unchanged, and are supplemented by a new relation symbol "$\in$".

Adopting the convention of Simpson [39], we let $\mathcal{L}_2$ denote the language of second order arithmetic. In the following, without explicit mention, we will let capital letters denote set variables while lower-case letters will denote number variables.

Definition 1 (Second order arithmetic). The axioms of second order arithmetic consist of the universal closure of the following $\mathcal{L}_2$-formulas.

1. Basic Axioms:
   - $n+1 \neq 0$
   - $m+1 = n+1 \rightarrow m = n$
   - $m+0 = m$
   - $m+(n+1) = (m+n)+1$
   - $m \cdot 0 = 0$
   - $m \cdot (n+1) = (m \cdot n)+m$
   - $\neg(m < 0)$
   - $m < n+1 \rightarrow (m < n \vee m = n)$
   - $\neg(n \in m)$
   - $\neg(X \in n)$
   - $\neg(X \in Y)$
2. Induction Axiom: $(0 \in X \wedge \forall n\, (n \in X \rightarrow n+1 \in X)) \rightarrow \forall n\, (n \in X)$
3. Comprehension Axioms:

$$\exists X\ \forall n\ (n \in X \leftrightarrow \varphi(n)),$$

   where $\varphi(n)$ is any $\mathcal{L}_2$-formula in which X does not occur freely.

In the context of reverse mathematics, in order to investigate the strength of different axiom systems, we need to first agree on a base system, that is, on the basic logical facts that we take for granted.

Definition 2 (Induction schemes). Given a set of formulae $\mathcal{B}$, the $\mathcal{B}$-induction scheme consists of all axioms of the form

$$(\varphi(0) \wedge \forall n\ (\varphi(n) \rightarrow \varphi(n+1)))\ \rightarrow\ \forall n\ (\varphi(n))$$

for any formula $\varphi(n) \in \mathcal{B}$ in which X does not occur freely.

Definition 3 (Base system RCA_0). RCA_0 is the subsystem of second order arithmetic consisting of the Basic Axioms as in Definition 1 (1), the Σ^0_1-induction scheme as in Definition 2, and the Comprehension Axioms as in Definition 1 (3) restricted to the class of Δ^0_1-formulas.

It is reasonable to use RCA_0 as base system for the investigation of stronger axiom systems in the context of reverse mathematics as it captures the effective aspects of mathematics. Additionally, it was shown that a fair number of mathematical theories can be developed relying solely on RCA_0; for details, see Simpson [39]. In this article we will also follow this established pratice unless otherwise stated.

It is common to informally refer to different base systems as different logical *principles*, and we will employ this expression frequently in the following.

3 Some second order combinatorial principles

A model of a set of second order arithmetical principles in general takes the form $\mathcal{M} = (M, S, +, \cdot, 0, 1)$ where M is the first order part of the structure and S is the second order part. If we decide not to require that all of the axioms of Definiton 1 hold, but only a subset of them, such as RCA_0, then it is not guaranteed anymore that a model of such an axiom set has $S = \mathcal{P}(M)$; typically, S will be much smaller. If $M = \omega$, $\mathcal{M}$ is called an ω-model, and S is a Turing ideal in this case.

The major textbook of reverse mathematics, Simpson [39], describes the five major axiom systems of reverse mathematics that cover many branches of mathematics, such as algebra, analysis, etc. Another recent textbook by Hirschfeldt [16] puts a particular focus on the role of Ramsey theory for reverse mathematics.

Before we can define the principles that we will study in this article, we need the following definitions.

Definition 4 (Weakly represented partial functions). A partial function f is said to be weakly represented by a set A if, for every x and y, there exists a z with $\langle x, y, z\rangle \in A$ iff

1. $x \in \mathrm{dom}(f) \wedge f(x) = y$ and (representation)
2. $\forall x, y, y', z, z'\ [(\langle x, y, z\rangle \in A \wedge \langle x, y', z'\rangle \in A)\ \rightarrow\ y = y']$ and (consistency)
3. $\forall x, y, y', z, z'\ [(\langle x, y, z\rangle \in A \wedge z < z')\ \rightarrow\ \langle x, y, z'\rangle \in A]$ and (monotonicity)

4. $\exists z\ \langle x, y, z\rangle \in A \ \rightarrow\ \forall t < x\ \exists y'\ \exists z'\ \langle t, y', z'\rangle \in A.$ (downward closure)

Definition 5 (Weakly represented families of functions). Let $A \in S$ be given and write $A_e = \{n\colon \langle e, n\rangle \in A\}$, $e \in M$, for its rows. For each e, write f_e for the (possibly partial) function weakly represented by A_e.

Then a set of total functions $\mathcal{F}$ is said to be a weakly represented family of functions represented by A if we have that $\mathcal{F}$ contains exactly those f_e, $e \in M$, that are total.

Note that all functions in a weakly represented family are by definition total. Rows A_e of A that do not represent such a function are ignored.

Definition 6 (Weakly represented families of sets). A set of sets $\mathcal{S}$ is said to be a weakly represented family of sets if their corresponding characteristic functions form a weakly represented family of functions.

Definition 7. $\mathcal{F}$ is said to be a uniform family of sets represented by A if

$$\mathcal{F} = \{A_e\colon e \in M\}$$

where $A_e = \{n\colon \langle e, n\rangle \in A\}$, $e \in M$.

Remark 8. It is easy to see that every uniform family of sets represented by some A is also a weakly represented family of sets represented by some B where $A =_{\mathrm{T}} B$.

One motivation for introducing weakly represented families is that the set of all partial recursive functions is a weakly represented family of functions. Similarly, it can easily be seen that in the classical setting the collection of all recursive sets is a weakly represented family of sets. This is because the class of characteristic functions of recursive sets

$$\mathcal{F} = \{\varphi_e\colon \varphi_e \text{ is total} \wedge \mathrm{range}(\varphi_e) \subseteq \{0, 1\}\}$$

can be weakly represented by a recursive set in any model of RCA_0.

These are examples of how the notion of weakly represented families enables us to talk about more and larger sets of functions; and this new ability then allows us to define new reverse mathematics principles, as we will now see.

Friedberg [13] constructed a maximal set, that is, an r.e. set A with infinite complement such that any other r.e. set B either contains almost all or almost none of the elements of the complement of A. As it turned out, the property of the complement being either almost contained in or being almost disjoint from every recursively enumerable set plays an important role in recursion theory, and thus it was given a name of its own, *cohesiveness*. This is a special case of the following more general definition.

Definition 9 (Cohesive set). For a set $A \subseteq M$, write $\bar{A}$ for $M \setminus A$. Then given a set of sets $\mathcal{F} \subseteq \{0, 1\}^M$, a set G is said to be $\mathcal{F}$-cohesive if for any $A \in \mathcal{F}$, either $G \subseteq^* A$ or $G \subseteq^* \bar{A}$. If $\mathcal{F}$ is the collection of all recursive sets, then G is called r-cohesive.

Statement 10 (Cohesion Principle COH). *For every uniform family $\mathcal{F}$ of sets, there exists an $\mathcal{F}$-cohesive set.*

While recursion theorists were originally interested in the degree-theoretic properties of cohesiveness, it turned out that it was relevant in reverse mathematics as well: Mileti [31] showed that Ramsey's Theorem for Pairs implies COH; and Cholak, Jockusch and Slaman [5] showed that Ramsey's Theorem for Pairs is equivalent to Stable Ramsey's Theorem for Pairs together with COH. For a detailed account of the role that COH has played in reverse mathematics, see Hirschfeldt [16].

In this article we will also study COHW, a variant of COH that takes advantage of the new possibilities introduced with the notion of weakly represented families of sets.

Statement 11 (Cohesion for weakly represented families COHW). *For every weakly represented family $\mathcal{F}$ of sets, there exists an $\mathcal{F}$-cohesive set.*

By Remark 8, COHW trivially implies COH. But we will show that the other implication does not hold, not even over ω-models.

Statement 12 (Domination Principle DOM). *Given any weakly represented family of functions $\mathcal{F}$, there exists a function g such that for every $f \in \mathcal{F}$ there is some $b \in M$ such that $g(x) > f(x)$ for all $x > b$.*

In a follow-up study to the present article, Hölzl, Jain and Stephan [19] establish further properties of DOM, including the following.

- Over RCA_0, $\mathsf{B}\Sigma_2 + \mathsf{DOM} \vdash \mathsf{I}\Sigma_2$;
- Over $\mathsf{RCA}_0 + \mathsf{DOM}$, the index set E of a weakly represented family is limit-recursive, that is, there is a binary $\{0,1\}$-valued function g such that for all $e \in M$, if $e \in E$ then $\exists s\, \forall t > s\, [g(e,t) = 1]$ else $\exists s\, \forall t > s\, [g(e,t) = 0]$. (Here, for a weakly represented family $\mathcal{F}$ of functions represented by A, we call the set of $e \in M$ for which f_e, as in Definition 5, is total, the *index set* of $\mathcal{F}$.)

We will show that over RCA_0 and $\mathsf{B}\Sigma_2$, DOM implies COH and COHW.

Statement 13 (Hyperimmunity Principle HI). *Given any weakly represented family of functions $\mathcal{F}$, there exits a function g such that for each $f \in \mathcal{F}$ and each $b \in M$ we have $g(x) > f(x)$ for some $x > b$.*

Note that HI is weaker than DOM. Hirschfeldt, Shore and Slaman [18] define the principle OPT, which they show [18, Theorem 5.7] to be equivalent to the statement that for every $f \in S$ there is a $g \in S$ such that f does not compute a function majorising g; thus this principle is equivalent to HI.

For $f, g \in M^M$ we write $f <^* g$ to express that $\{n \in M : g(n) \leq f(n)\}$ is finite. The symbol "$\leq^*$" is defined accordingly. A subset $\mathcal{F} \subseteq M^M$ is called *bounded* if there exists $g \in M^M$ such that for all $f \in \mathcal{F}$ we have $f <^* g$. Otherwise $\mathcal{F}$ is said to be *unbounded*.

Statement 14 (Meeting Principle MEET). *Given any weakly represented family of functions $\mathcal{F}$, there exits a function g such that for each $f \in \mathcal{F}$ the set $\{n \in M : f(n) = g(n)\}$ is infinite.*

We will show that HI and MEET are equivalent.

Definition 15. We say that a function g avoids a function f if

$$\{n \in M : f(n) = g(n)\}$$

is finite.

Statement 16 (Avoidance Principle AVOID). *Given any weakly represented family of functions $\mathcal{F}$, there exits a function g avoiding all $f \in \mathcal{F}$.*

Two subsets A and B of M are said to be *almost disjoint* if $A \cap B$ is finite. A set $\mathcal{F} \subseteq \{0,1\}^M$ is called *almost disjoint* if any two distinct elements of $\mathcal{F}$ are almost disjoint. A set $\mathcal{F} \subseteq \{0,1\}^M$ is called *maximal almost disjoint* if it is infinite and almost disjoint and is not properly contained in any larger almost disjoint set. Formalising that a family is infinite is somewhat tricky; we use the following approach.

Definition 17. We call a weakly represented family $\mathcal{F}$ finite if there is a weakly represented family $\mathscr{G}$ with finite index set such that $\mathcal{F} = \mathscr{G}$. Otherwise we call $\mathcal{F}$ infinite.

Statement 18 (Maximal Almost Disjoint Family Principle MAD). *There exists a weakly represented family $\mathcal{F}$ of infinite sets such that the following three conditions hold:*

- *$\mathcal{F}$ is infinite;*
- *if $A, B \in \mathcal{F}$ are pairwise different, then $A \cap B$ is finite;*
- *for every infinite set $C \in S$ there is a $D \in \mathcal{F}$ such that $C \cap D$ is infinite.*

For a set $A \subseteq M$, let us temporarily write A^0 for A and A^1 for $\bar{A}$. A family $\mathcal{F} \subseteq \mathcal{P}(M)$ is said to be *independent* if for any $n \geq 1$, any collection $\{A_0, \dots, A_{n-1}\} \subseteq \mathcal{F}$, and any string $\sigma \in 2^n$, the set $\bigcap_{i<n} A_i^{\sigma(i)}$ is infinite. A *maximal independent* family is an independent family that can not be extended to a strictly larger independent family.

Statement 19 (Maximal Independent Family Principle MIND). *There exists a weakly represented family of infinite sets that is maximal independent.*

Statement 20 (Biimmunity Principle BI). *For every weakly represented family $\mathcal{F}$ of infinite sets there is a set $B \in S$ such that there is no set $A \in \mathcal{F}$ with $A \subseteq B$ or $A \subseteq \overline{B}$.*

4 Cardinal invariants

We now discuss the nine cardinal invariants of the continuum that are considered in this paper, the most basic being the cardinality of the continuum.

Definition 21. $\mathfrak{c} = 2^{\aleph_0} = |\mathbb{R}|$.

Recall that the Continuum Hypothesis CH is the statement that $\mathfrak{c} = \aleph_1$. The analogue of CH in a model $\mathcal{M} = (M, S, \cdot, +, 0, 1)$ is the statement that there is a weakly represented family of sets represented by $A \in S$ such that the characteristic function of every element of S appears in A. In other words, this states that there is a set in S which "encodes in a nice way" all the subsets of M that can be "seen by" $\mathcal{M}$. The simplest example of this is the case where S consists exactly of the recursive sets.

Recall the partial order $\langle M^M, <^* \rangle$ defined in the previous section. We assume that ZFC is our base theory when talking about cardinal invariants of the continuum. Therefore, we only consider the restriction of this partial order to ω^ω in this section. So for $f, g \in \omega^\omega$, $g <^* f$ means that $\{n \in \omega : g(n) \geq f(n)\}$ is finite; and "finite" here does not mean finite in the sense of some specific model of second order arithmetic, but finite as defined within ZFC. Recall that a family $\mathcal{F} \subseteq \omega^\omega$ is *unbounded* if there is no $g \in \omega^\omega$ such that $\forall f \in \mathcal{F}\ [f <^* g]$ and $\mathcal{F}$ is *dominating* if for all $g \in \omega^\omega$ there exists an $f \in \mathcal{F}$ with $g <^* f$. It is clear that every dominating set is unbounded. Based on these definitions, we define the following two cardinal invariants.

Definition 22.
$$\mathfrak{b} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega \wedge \mathcal{F} \text{ is unbounded}\}$$
$$\mathfrak{d} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega \wedge \mathcal{F} \text{ is dominating}\}$$

It is easy to prove that $\aleph_1 \leq \mathrm{cf}(\mathfrak{b}) = \mathfrak{b} \leq \mathrm{cf}(\mathfrak{d}) \leq \mathfrak{d} \leq \mathfrak{c}$, where $\mathrm{cf}(\kappa)$ denotes the cofinality of the cardinal κ. It is also a classical theorem of Hechler [14] that these are the only restrictions that are provable in ZFC. In keeping with the intuition described in the introduction, in a second order model $\mathcal{M} = (M, S, \cdot, +, 0, 1)$, the statement $\mathfrak{b} = \aleph_1$ should correspond to the statement that there exists a set in S which "nicely encodes" an unbounded family of functions from the point of view of $\mathcal{M}$. In other words, $\mathfrak{b} = \aleph_1$ should correspond to the statement that there is a weakly represented family of functions $\mathcal{F}$ represented by some $A \in S$ such that no function in S dominates, in the sense of the partial order $\langle M^M, <^* \rangle$, all the elements of $\mathcal{F}$. This is the negation of the principle DOM. So DOM is the analogue of $\mathfrak{b} > \aleph_1$. Similarly HI corresponds to $\mathfrak{d} > \aleph_1$.

Another important pair of cardinals come from the notion of splitting. Recall that for a set X and a cardinal κ, $[X]^\kappa = \{A \subseteq X \colon |A| = \kappa\}$, in particular $[\omega]^\omega$ denotes the set of infinite subsets of ω. Let $A, B \subseteq \omega$. We say that A *splits* B if both $B \cap A$ and $B \cap \bar{A}$ are infinite. A set $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is called a *splitting family* if $\forall B \in [\omega]^\omega\ \exists A \in \mathcal{F}\ [A \text{ splits } B]$. A set $A \subseteq \omega$ is said to *reap* a family $\mathcal{F} \subseteq \mathcal{P}(\omega)$ if for all $B \in \mathcal{F}$ we have that A splits B. A family $\mathcal{F} \subseteq [\omega]^\omega$ is *unreaped* if there is no $A \in \mathcal{P}(\omega)$ which reaps $\mathcal{F}$. The following cardinals correspond to the notions of splitting and reaping.

Definition 23. $\mathfrak{s} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{P}(\omega) \wedge \mathcal{F} \text{ is a splitting family}\}$

$$\mathfrak{r} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq [\omega]^{\omega} \wedge \mathcal{F} \text{ is an unreaped family}\}$$

It is not difficut to prove that $\mathfrak{s} \leq \mathfrak{d}$, and this proof dualizes to show that $\mathfrak{b} \leq \mathfrak{r}$ (see Blass [2]). Blass and Shelah constructed a model with $\aleph_1 = \mathfrak{r} < \mathfrak{s} = \aleph_2$ (see Bartoszyński and Judah [1, Section 7.4.D]) and $\aleph_1 = \mathfrak{s} < \mathfrak{b} = \aleph_2$ holds in the Laver model (see Bartoszyński and Judah [1, Section 7.3.D]). The notion of a cohesive set in recursion theory is related to the notion of splitting. To say that G is $\mathcal{F}$-cohesive is the same as saying that G is not split by any member of $\mathcal{F}$. So the principle COHW corresponds to the statement $\mathfrak{s} > \aleph_1$ because it says that no weakly represented family $\mathcal{F} \in S$ has the property that every $A \in S$ is split by some member of $\mathcal{F}$ — in other words, S does not "nicely encode" any splitting family in the sense of $\mathcal{M}$. The principle COH is related to COHW and satisfies COHW $\vdash$ COH properly; there is no direct analogue of COH in set theory. The principle BI corresponds to the statement $\mathfrak{r} > \aleph_1$ and the reverse mathematical analogue of the ZFC theorem $\mathfrak{s} \leq \mathfrak{d}$ is the statement that COHW implies HI. However in parallel with Blass and Shelah's result that the statement $\aleph_1 = \mathfrak{b} = \mathfrak{r} < \mathfrak{s} = \aleph_2$ is consistent with ZFC, it holds that COHW does not imply DOM; the full result has no analogue as COHW implies HI and HI implies BI. Furthermore, the implication DOM $\vdash$ HI has the analogue $\mathfrak{b} \leq \mathfrak{d}$ in ZFC. In both cases, the inverse implication does not hold.

The next group of cardinals that we define stem from the context of categoricity. Recall that a set $X \subseteq \mathbb{R}$ is called *nowhere dense* if the interior of its closure is empty. A subset of $\mathbb{R}$ is *meager* if it is the union of countably many nowhere dense sets. We define the following cardinals.

Definition 24. $\mathrm{cov}(\mathcal{C}) = \min\left\{|\mathcal{F}| : \begin{array}{c}\mathcal{F} \text{ consists of meager subsets of } \mathbb{R} \\ \text{and } \bigcup \mathcal{F} = \mathbb{R}\end{array}\right\}$

$$\mathrm{non}(\mathcal{C}) = \min\{|A| : A \text{ is a non-meager subset of } \mathbb{R}\}$$

Here $\mathcal{C}$ stands for category. These topologically defined cardinals have purely combinatorial characterizations, as the following theorem shows.

Theorem 25 (Miller[32]).

1. $\mathrm{cov}(\mathcal{C})$ *is the minimal cardinal* κ *such that there exists an* $\mathcal{F} \subseteq \omega^{\omega}$ *with* $|\mathcal{F}| = \kappa$ *and such that for all* $g \in \omega^{\omega}$ *there is an* $f \in \mathcal{F}$ *such that*
$$\{n \in \omega\colon f(n) = g(n)\}$$
is finite.
2. $\mathrm{non}(\mathcal{C})$ *is the minimal cardinal* κ *such that there exists an* $\mathcal{F} \subseteq \omega^{\omega}$ *with* $|\mathcal{F}| = \kappa$ *and such that for all* $g \in \omega^{\omega}$ *there is an* $f \in \mathcal{F}$ *such that*
$$\{n \in \omega\colon g(n) = f(n)\}$$
is infinite.

We remind the reader that in the above theorem "finite" and "infinite" are not to be understood in the sense of $\mathcal{M}$, but as those terms as defined within ZFC.

The above theorem allows us to formulate analogues of these topological invariants in any model of second order arithmetic. For $\operatorname{cov}(\mathcal{C})$ to be "small" in a second order model $\mathcal{M} = (M, S, \cdot, +, 0, 1)$, we would like to have a weakly represented family of functions $\mathcal{F} \in S$ with the property that for any function $g \in S$, $\{n \in M : f(n) = g(n)\}$ is finite in the sense of $\mathcal{M}$. The principle MEET says that no such weakly represented family exists. Thus MEET corresponds to the statement that $\operatorname{cov}(\mathcal{C}) > \aleph_1$. Similarly, AVOID is the analogue of $\operatorname{non}(\mathcal{C}) > \aleph_1$. As it is easy to prove in ZFC that $\operatorname{cov}(\mathcal{C}) \leq \mathfrak{d}$, one would expect MEET to imply HI, and indeed this is easy to check. But somewhat unexpectedly we will prove that MEET and HI are equivalent — at least for ω-models. This contrasts with the fact that $\operatorname{cov}(\mathcal{C}) = \aleph_1 < \aleph_2 = \mathfrak{b} = \mathfrak{d}$ holds in the Laver model (see Bartoszyński and Judah [1, Section 7.3.D]). As a result, in the classical ZFC context, we do not even have that DOM implies MEET. Dualizing the equivalence of MEET and HI one would expect AVOID to be equivalent to DOM. Indeed, DOM implies AVOID by definition; however, we show in Theorem 41 that AVOID does not imply HI, and therefore not DOM. Also it is consistent that $\mathfrak{b} = \aleph_1 < \aleph_2 = \operatorname{cov}(\mathcal{C})$; in fact it is folklore that this holds in the Cohen model. This is reflected by the fact that MEET does not imply DOM, which follows immediately from Theorem 38. Next, regarding $\operatorname{non}(\mathcal{C})$, it is easy to see by Theorem 25 (2), that $\mathfrak{b} \leq \operatorname{non}(\mathcal{C})$ holds in ZFC, and, accordingly, DOM implies AVOID. It is also easy to prove in ZFC that $\mathfrak{s} \leq \operatorname{non}(\mathcal{C})$. This is only partially true in the reverse mathematical context. Namely, we will prove that COHW implies AVOID in ω-models. However this is not true in all non-ω-models, as we will show. Finally, in the classical ZFC context, $\mathfrak{d}$ and $\operatorname{non}(\mathcal{C})$ are independent, meaning that while it is consistent to have $\aleph_1 = \mathfrak{d} < \operatorname{non}(\mathcal{C}) = \aleph_2$ (see Bartoszyński and Judah [1, Section 7.3.B]) it is also consistent to have $\aleph_1 = \operatorname{non}(\mathcal{C}) < \mathfrak{d} = \aleph_2$ (see Bartoszyński and Judah [1, Section 7.3.E]). This is reflected by the independence of AVOID and MEET, even in ω-models.

We also considered cardinal invariants associated with almost disjointness and independence. In the ZFC context, $A, B \subseteq \omega$ are said to be *almost disjoint* if $|A \cap B| < \aleph_0$. A family $\mathscr{A} \subseteq [\omega]^\omega$ is *almost disjoint* if its members are pairwise almost disjoint. An infinite almost disjoint family $\mathscr{A}$ is said to be *maximal almost disjoint* if it is not properly contained in any larger almost disjoint family. Any infinite almost disjoint set can be extended to a maximal almost disjoint set by Zorn's Lemma.

Similarly a family $\mathcal{F} \subset [\omega]^\omega$ is called *independent* if for each $n \geq 1$, each collection $\{A_0, \ldots, A_{n-1}\} \subset \mathcal{F}$, and each string $\sigma \in 2^n$, $\left|\bigcap_{i<n} A_i^{\sigma(i)}\right| = \aleph_0$, where A_i^0 is A_i and A_i^1 is $\bar{A}_i$. A *maximal independent family* is an independent family $\mathcal{F} \subset [\omega]^\omega$ which is not properly contained in any larger independent family. Zorn's Lemma also guarantees the existence of maximal independent families.

Observe that a second order model $\mathcal{M} = (M, S, \cdot, +, 0, 1)$ need not satisfy any principle akin to Zorn's lemma. Therefore there need not be any maximal almost disjoint or maximal independent families in S, weakly represented or otherwise.

Definition 26. $\mathfrak{a} = \min\{|\mathscr{A}| : \mathscr{A} \subseteq [\omega]^{\omega}$ is a maximal almost disjoint family$\}$
$\mathfrak{i} = \min\{|\mathscr{A}| : \mathscr{A} \subseteq [\omega]^{\omega}$is a maximal independent family$\}$

The principle MAD says that there is a weakly respresented maximal almost disjoint family and so it corresponds to $\mathfrak{a} = \aleph_1$. Similarly MIND corresponds to $\mathfrak{i} = \aleph_1$.

We prove that, at least for ω-models, MAD holds iff DOM fails. Since in ZFC the inequality $\mathfrak{b} \leq \mathfrak{a}$ holds by folklore, it does not come as a surprise that DOM implies ¬MAD. However, $\aleph_1 = \mathfrak{b} < \mathfrak{a} = \aleph_2$ is consistent by a theorem of Shelah [38], so the fact that ¬DOM implies MAD is unexpected.

Similarly, we prove that, at least for ω-models, MIND holds iff BI fails. One can easily prove in ZFC that $\mathfrak{r} \leq \mathfrak{i}$ and so the direction from BI to ¬MIND is unsurprising. But once again the consistency of $\mathfrak{r} = \aleph_1 < \aleph_2 = \mathfrak{i}$ was proved by Blass and Shelah (see Bartoszyński and Judah [1, Section 7.4.D]), making the implication from ¬BI to MIND unexpected.

5 Cohesion Principles

In the following we will introduce definitions needed for this paper; for the recursion-theoretic background, the reader is referred to the textbooks of Downey and Hirschfeldt [10], Nies [34], Odifreddi [35,36], Simpson [39] and Soare [41].

Definition 27. Let A and B be sets. A is PA-complete with respect to B (written as $A \gg B$) if for every partial B-recursive $\{0, 1\}$-valued function f, there exists an A-recursive total extension g of f. In this definition we can replace sets by degrees in the canonical way.

Definition 28. Let A and B be sets. A is hyperimmune-free with respect to B if every function recursive in $A \oplus B$ is dominated by some B-recursive function.

Theorem 29. *Over* RCA_0, COH *does not imply* COHW*. This even holds for ω-models.*

To proof the non-implication for ω-models, we need the following lemmata and theorem. The first lemma establishes a relationship between two 1-generic sets and their join. It is the genericity analogue of van Lambalgen's Theorem 40.

Lemma 30 (Yu [43]). *The following are equivalent for $n \geq 1$.*

1. *$A \oplus B$ is n-generic;*
2. *A is n-generic and B is n-generic relative to A;*
3. *B is n-generic and A is n-generic relative to B.*

The following theorem is a reformulation and slight variation of a result of Jockusch and Stephan [24, Theorem 2.1]; the proof is largely identical and omitted here.

Theorem 31. *Let $\mathcal{F}$ be a uniform family represented by A. If $B' \gg A'$ then there is a B-recursive $\mathcal{F}$-cohesive set.*

Lemma 32. *There is a sequence of sets $(A_i : i \in \omega)$ such that, for every $i \in \omega$, A_i is 1-generic and not high, $A_{i+1} \geq_{\mathrm{T}} A_i$ and $A'_{i+1} \gg A'_i$.*

Proof. Let $B_0 = \emptyset'$. If, for some $i \in \omega$, B_i with $B'_i \leq_{\mathrm{T}} \emptyset''$ has been inductively defined, then we compute relative to B_i a tree T each path of which is a complete extension of PA relative to B_i, see Odifreddi [35]. Then by Jockusch and Soare's Low Basis Theorem [23] relative to B_i we have a path $B_{i+1} \in [T]$ such that $B_{i+1} \gg B_i$, $B_{i+1} \geq_{\mathrm{T}} B_i$ and such that B_{i+1} is low relative to B_i, that is, $B'_{i+1} \leq_{\mathrm{T}} B'_{i+1} \oplus B_i \leq_{\mathrm{T}} B'_i \leq_{\mathrm{T}} \emptyset''$.

It is well-known that the sets provided by Friedberg's Jump Inversion Theorem [12] can be assumed to be 1-generic; see, for example, Stephan [42, Theorem 5.4]. By applying this result to B_0 we obtain a 1-generic set A_0 such that $A'_0 = B_0$ (that is, A_0 is low). With A_i defined for some $i \in \omega$, and using that $B_{i+1} \geq_{\mathrm{T}} B_i = A'_i$, we can apply the Jump Inversion Theorem relative to A_i to the set B_{i+1} to obtain a set C_{i+1} with $C'_{i+1} = B_{i+1}$ and C_{i+1} being 1-generic relative to A_i. We let $A_{i+1} = C_{i+1} \oplus A_i$ and by Lemma 30 and using that A_i is 1-generic by induction hypothesis we again have that A_{i+1} is 1-generic.

Note that for all $i \in \omega$ we have $A''_i = B'_i \leq_{\mathrm{T}} \emptyset''$ and thus $A'_i <_{\mathrm{T}} \emptyset''$. It follows that A_i is not high. □

Now let $S = \{A \subseteq \omega : A \leq_{\mathrm{T}} A_0 \oplus \ldots \oplus A_n \text{ for some } n \in \omega\}$, where the sets A_i, $i \in \omega$, are as in Lemma 32. The following lemmata show that the ω-model $\mathcal{M} = (\omega, S, +, \cdot, 0, 1)$ satisfies COH and RCA_0, but not COHW.

Lemma 33. $\mathcal{M} \models \mathsf{COH} + \mathsf{RCA}_0$

Proof. First note that S is closed under Turing reducibility and join, thus $\mathcal{M}$ is a model of RCA_0.

To see that $\mathcal{M}$ is also a model of COH, let a uniform familiy $\mathcal{F}$ represented by $A \leq_{\mathrm{T}} A_0 \oplus \ldots \oplus A_n =_{\mathrm{T}} A_n$ for large enough n be given. Then, by construction, $A'_{n+1} \gg A'_n$, and we can apply Theorem 31 with A_{n+1} substituted for B and A_n substituted for A to see that there is an A_{n+1}-recursive $\mathcal{F}$-cohesive set. □

Corollary 34. $\mathcal{M} \not\models \mathsf{COHW}$

Proof. By Jockusch and Stephan [24, Theorem 2.9] each cohesive 1-generic degree is high. But by Lemma 32, no set in S is high, therefore none of the 1-generic sets A_i, $i \in \omega$, is cohesive. Since by Jockusch [22] the cohesive degrees are upwards closed this implies that no set in S is cohesive. Now by Jockusch and Stephan [24, Corollary 2.4], the cohesive and the r-cohesive degrees coincide, so no set in S is r-cohesive. In particular, if $\mathcal{F}$ is the weakly represented family consisting of all recursive sets, there exists no $\mathcal{F}$-cohesive set in S. So COHW fails to hold. □

This concludes the proof of Theorem 29, separating COH and COHW over RCA_0.

Theorem 35. COH *does not imply* AVOID. *This even holds for ω-models.*

We use the following well-known lemma.

Lemma 36 (Demuth and Kučera [9]). *No* 1*-generic set computes a diagonally non-recursive function.*

Proof (Theorem 35). We again use the above ω-model $\mathcal{M}$ with second order part S. Observe that S is a downward closure of non-high 1-generic sets. In particular, by Lemma 36, all sets in S are neither diagonally non-recursive nor high. By Kjos-Hanssen, Merkle, and Stephan [25, Theorem 5.1 ($\neg(3) \Rightarrow \neg(1)$)], this implies that no $A \in S$ computes a function avoiding all total recursive functions. As the set of all total recursive functions is a weakly represented family, this contradicts AVOID. □

The next theorem illustrates once more the difference in reverse mathematics strength between COH and COHW.

Theorem 37. COHW $\vdash$ AVOID *for ω-models.*

More precisely, given any r-cohesive set G, one can recursively produce a total function g such that $\{n \in \omega \colon g(n) = \varphi_e(n)\}$ is finite for every total recursive function φ_e.

Proof. Let $\mathcal{F}$ be the collection of all total recursive functions. We will show that there exists a function $g \in S$ such that for every $f \in \mathcal{F}$ we have that $\{n \in \omega \colon f(n) = g(n)\}$ is finite. Then the general case follows by relativization.

Let $\mathcal{F}'$ be the collection of all recursive sets. COHW ensures the existence of an $\mathcal{F}'$-cohesive set, say G. If G is high, then by Martin [30], there exists a function g recursive in G that dominates every total recursive function, and we are done.

If G is not high, then by Jockusch and Stephan [24] there exists an effectively immune set A recursive in G. Here we call A effectively immune if there is a recursive function p such that for any r.e. set W_e we have $W_e \subseteq A \to |W_e| < p(e)$. Fix this p and assume without loss of generality that it is increasing.

Let f be the total recursive function such that

$$W_{f(e,i)} = \begin{cases} W_{\varphi_i(e)} & \text{if } \varphi_i(e)\downarrow, \\ \emptyset & \text{otherwise.} \end{cases}$$

Let g be the total recursive function such that $W_{g(e)}$ consists of the first

$$p(\max\{f(i,e) \colon i \leq e\}) + 1$$

elements of A.

Claim. For all $i \leq e$ we have $g(e) \neq \varphi_i(e)$.

Proof. Suppose otherwise, then $g(e) = \varphi_i(e)$ for some $i \leq e$. Then $\varphi_i(e)\downarrow$ and $W_{f(e,i)} = W_{\varphi_i(e)} = W_{g(e)} \subseteq A$, so

$$p(f(e,i)) < p(\max\{f(e,j)\colon j \leq e\}) + 1 = |W_{g(e)}| = |W_{f(e,i)}| < p(f(e,i)),$$

which is a contradiction. ◇

Then g is the required function. □

Given that the previous proof was carried out in the standard model, it is natural to ask how COHW interacts with AVOID in non-standard models.

6 The Meeting and Hyperimmunity Principles

In this section we investigate the principles MEET and HI and their relations to each other as well as to other principles.

Theorem 38. *Over* RCA_0*,* MEET *and* HI *are equivalent.*

Proof. MEET $\vdash$ HI: If g is as in the statement of MEET, then HI holds with $g+1$ substituted for g.

HI $\vdash$ MEET: Let an arbitrary model $\mathcal{M} = (M, S, +, \cdot, 0, 1)$ be given. Let $\mathcal{F}$ be a weakly represented family represented by A and let A_e and f_e, for $e \in M$, be as in Definition 5. Define a function $\widetilde{f_e}$ via $x \mapsto n_x$ for $x \in M$, where n_x is the first number of the form $\langle\langle e, x\rangle, y, z\rangle$ inside A_e, if it exists; note that $y = f_e(\langle e, x\rangle)$ whenever n_x exists. Then $\widetilde{f_e}$ is total iff f_e is total.

Note that the set $\mathcal{F}' = \{\widetilde{f_e}\colon \widetilde{f_e} \text{ is total}\}$ is again a weakly represented family. By applying HI to $\mathcal{F}'$ we obtain a function $\widetilde{g}$ such that for each total $\widetilde{f_e}$ there are infinitely many x with $\widetilde{g}(x) > \widetilde{f_e}(x)$.

Then define $g(\langle e, x\rangle)$ as follows: If there is a number m of the form $\langle\langle e, x\rangle, y, z\rangle$ in A_e such that $m < \widetilde{g}(x)$, then let $g(\langle e, x\rangle) = y$, else $g(\langle e, x\rangle) = 0$. The function g is in S and is total; furthermore, whenever $\widetilde{g}(x) > \widetilde{f_e}(x)$ then we have $g(\langle e, x\rangle) = f_e(\langle e, x\rangle)$ and thus for all total f_e there are infinitely many n with $g(n) = f(n)$. This implies MEET. □

Our next result shows that AVOID is incomparable with HI. As essential tools we employ the following two well-known results; to see the first, apply the hyperimmune-free basis theorem of Jockusch and Soare [23] to the complement of the first component of the universal Martin-Löf test.

Lemma 39. *There exists a hyperimmune-free Martin-Löf random set.*

Theorem 40 (van Lambalgen [29]). *The following are equivalent.*

1. *$A \oplus B$ is n-random.*
2. *A is n-random and B is n-random relative to A.*
3. *B is n-random and A is n-random relative to B.*

Theorem 41. AVOID *does not imply* HI. *This even holds for* ω*-models.*

Proof. Let A be a hyperimmune-free Martin-Löf random, as in Lemma 39. For $i \in \omega$, let $A_i = \{x \colon \langle i, x\rangle \in A\}$. Then by Theorem 40, for every $i \in \omega$, A_{i+1} is Martin-Löf random relative to $A_0 \oplus \ldots \oplus A_i$. Fix the model $\mathcal{M} = (\omega, S, +, \cdot, 0, 1)$ with second order part $S = \{B \subseteq \omega \colon B \leq_{\mathrm{T}} A_0 \oplus \ldots \oplus A_i \text{ for some } i \in \omega\}$.

Let a weakly represented family $\mathcal{F}$ represented by $B \leq_{\mathrm{T}} A_0 \oplus \ldots \oplus A_i$ with $i \in \omega$ large enough be given. Fix a computably bijective map $\nu \colon \{0,1\}^* \to \omega$, and let g be the function $n \mapsto \nu(A_{i+1}(0) \ldots A_{i+1}(n))$. Fix any $f \in \mathcal{F}$; trivially, $f \leq_{\mathrm{T}} B$. Assume that g does not avoid f. Then there are infinitely many n such that the Kolmogorov complexity relative to B of $A_{i+1}(0) \ldots A_{i+1}(n)$ is less than $2\log(n)$, which contradicts that A_{n+1} is Martin-Löf random relative to $A_0 \oplus \ldots \oplus A_i$. Therefore $g \leq_{\mathrm{T}} A_{i+1} \in S$ is a function as required by AVOID.

On the other hand, for every $C \in S$ we have $C \leq_{\mathrm{T}} A$, and since A is hyperimmune-free, C is hyperimmune-free as well. As C was arbitrary, this implies that $\mathcal{M}$ does not satisfy HI. □

We now turn to the other direction.

Theorem 42. HI *does not imply* AVOID.

Proof. We again use the ω-model $\mathcal{M}$ from the proof of Theorem 35. As shown there, $\mathcal{M}$ does not satisfy AVOID.

To see that $\mathcal{M}$ satisfies HI, let a weakly represented family $\mathcal{F}$ represented by $A \leq_{\mathrm{T}} A_n$ be given. As A_{n+1} is by construction 1-generic relative to A_n, it is in particular hyperimmune relative to A. Then it computes a function g that is infinitely often larger than any function $f \leq_{\mathrm{T}} A$, and in particular g is for $\mathcal{F}$ as required by HI. □

A further interesting result is the following, which is in line with the fact from recursion theory that the Turing degrees of cohesive sets are hyperimmune.

Theorem 43. COHW *implies* HI.

Proof. Let $\mathcal{F}$ be a weakly represented family of functions represented by A, let f_e be as in Definition 5 and let $E = \{e \colon f_e \text{ is total}\}$.

Define for each $e \in M$ inductively a function g_e such that $g_e(0) = 1$ and $g_e(x+1) = \max\{f_e(x') + 1 \colon x' \leq g_e(x) + 1\}$. Next define for each $e \in M$ the set $B_e = \{y \colon \exists x\, [g_e(2x) \leq y < g_e(2x+1)]\}$. It is easy to see that $\mathcal{F}' = \{B_e \colon e \in E\}$ is a weakly represented family of sets.

By COHW there is an $\mathcal{F}'$-cohesive set $C \in S$. Then let h be the principal function of C; that is, h is strictly monotonically increasing and $C = \{h(0), h(1), \ldots\}$. Then $h \in S$ as well. Also note that $h(n) \geq n$ holds trivially for all $n \in M$.

Fix $e \in E$. Firstly, consider the case that there is some $b \in M$ such that $C \cap \{x \colon x \geq b\} \subseteq B_e$. Then $C \cap \{y : g_e(2x+1) \leq y < g_e(2x+2)\}$ is empty for almost all x. We claim that $h(g_e(2x+1)) > f_e(g_e(2x+1))$ for sufficiently large numbers x. To see this observe that $h(g_e(2x+1)) \in C$ and thus in B_e, which implies $h(g_e(2x+1)) \geq g_e(2x+2)$ as the smallest element of B_e larger than $g_e(2x+1)$ is $g_e(2x+2)$. Then, by definition of g_e,

$$\begin{aligned} h(g_e(2x+1)) &\geq g_e(2x+2) \\ &= \max\{f_e(x')+1 \colon x' \leq g_e(2x+1)+1\} \\ &> f_e(g_e(2x+1)). \end{aligned}$$

Secondly, consider the case that there is some $b \in M$ with the property that $(C \cap \{x \colon x \geq b\}) \cap B_e = \emptyset$. Then $C \cap \{y : g_e(2x) \leq y < g_e(2x+1)\}$ is empty for almost all x. For similar reasons as in the previous case, $h(g_e(2x)) > f_e(g_e(2x))$ for almost all x.

Due to C's $\mathcal{F}'$-cohesiveness, one of the two cases must occur. As a result, the set $\{y \in M \colon h(y) > f_e(y)\}$ is guaranteed to be infinite. Since $e \in E$ was chosen arbitrarily, the requirements of HI with regards to $\mathcal{F}$ are satisfied; furthermore, since $\mathcal{F}$ was chosen arbitrarily as well, HI holds in general. □

7 The Domination Principle

In this section we show that over $\mathsf{RCA}_0 + \mathsf{B}\Sigma_2$, the principle DOM implies COH and COHW. It is an open question whether the assumption $\mathsf{B}\Sigma_2$ is needed.

Theorem 44. *Over* $\mathsf{RCA}_0 + \mathsf{B}\Sigma_2$*,* DOM *implies* COH*.*

Proof. Hölzl, Jain and Stephan [19, Theorem 20] showed that over $\mathsf{RCA}_0 + \mathsf{B}\Sigma_2$, DOM implies $\mathsf{I}\Sigma_2$. Thus we can assume that $\mathsf{I}\Sigma_2$ holds for the purposes of this proof.

Let $\mathcal{M} = (M, S, +, \cdot, 0, 1)$ be a model of DOM and let $\mathcal{F}$ be a uniform family of sets represented by $A \in S$. For $e \in M$ let A_e be as in Definition 7 and let $\widetilde{f}_{e,x}(y)$ be the first $z > y$ with $\forall d \leq e\, [A_d(z) = A_d(x)]$ and let $\widetilde{\mathcal{F}}$ be the weakly represented family of those $\widetilde{f}_{e,x}$ which are total. By DOM there is a function $g \in S$ which dominates all members of $\widetilde{\mathcal{F}}$. Define an infinite set $G = \{x_0, x_1, \ldots\} \in S$ as follows:

- $x_0 = 0$ and
- Let $X_n = \{x_n+1, x_n+2, \ldots, x_n+g(x_n)\}$ and define x_{n+1} as the minimal $y \in X_n$ such that

$$A_0(y)A_1(y)\ldots A_{x_n}(y) = \max\{A_0(z)A_1(z)\ldots A_{x_n}(z) \colon z \in X_n\},$$

  where the maximum is with respect to $\leq_{\text{lex}}$, the lexigraphic ordering on strings.

Let $\Psi(e,x)$ be the statement

$$x \in G \;\wedge\; \forall y \geq x\, [y \in G \rightarrow A_0(y)A_1(y)\ldots A_{e-1}(y) = A_0(x)A_1(x)\ldots A_{e-1}(x)].$$

Claim. For all e, $\exists x\, (\Psi(e,x))$ holds.

Proof. As $\exists x\ \Psi(e,x)$ is a Σ^0_2-statement, using $\mathsf{I}\Sigma_2$, we can prove it by induction over $e \in M$.

The statement $\Psi(0,x)$ holds vacuously for all $x \in G$. So assume by induction that for a given $e \in M$, $\Psi(e,x')$ is true for some $x' \in G$. We distinguish two cases:

Case 1. $G \cap A_e$ is finite. Then there exists an $x'' \geq x'$ with $x'' \in G$ and $x'' > \max(A_e \cap G)$. Then for all $y \in G$ with $y \geq x''$, we have $A_e(y) = A_e(x'') = 0$ on the one hand; and by the induction hypothesis

$$A_0(y)A_1(y)\ldots A_{e-1}(y) = A_0(x'')A_1(x'')\ldots A_{e-1}(x'')$$

on the other hand. Thus $\Psi(e+1,x'')$ holds and $\exists x\ \Psi(e+1,x)$ is satisfied.

Case 2. $G \cap A_e$ is infinite. Then let x'' be any element of $G \cap A_e$ with $x'' \geq x'$. For all such x'' the function $\widetilde{f}_{e,x''}$ is the same and thus one can, without loss of generality, assume that x'' is large enough that $g(y) > \widetilde{f}_{e,x''}(y)$ for all $y \geq x''$ and $x'' > e+1$. Now let $n \in M$ be arbitrary such that $x_n \geq x''$. Then

$$A_0(x_{n+1})A_1(x_{n+1})\ldots A_{x_n}(x_{n+1}) \geq_{\text{lex}} A_0(x'')A_1(x'')\ldots A_e(x'')0^{x_n-e-1};$$

and thus,

$$A_0(x_{n+1})A_1(x_{n+1})\ldots A_e(x_{n+1}) = A_0(x'')A_1(x'')\ldots A_e(x'').$$

As $n \in M$ was arbitrary with $x_n \geq x''$ it follows that $\Psi(e+1,x'')$ holds and that $\exists x\ \Psi(e+1,x)$ is satisfied. ◇

Thus $\exists x\ \Psi(e,x)$ holds for all e, and in particular for each e there is an $x \in G$ with $A_e(y) = A_e(x)$ for all $y \geq x$ with $y \in G$. Thus COH is satisfied. □

In fact, we can obtain the following stronger result.

Corollary 45. *Over* $\mathsf{RCA}_0 + \mathsf{B}\Sigma_2$*,* DOM *implies* COHW*.*

This corollary follows immediately from Theorem 44 and the following observation.

Proposition 46. *Over* $\mathsf{RCA}_0 + \mathsf{DOM}$*,* COH *implies* COHW*.*

Proof. Let $\mathcal{F}$ be a weakly represented family of sets represented by A, let f_e be as in Definition 5, let $E = \{e \colon f_e \text{ is total}\}$ and for all $e \in E$ write B_e for the set whose characteristic function is f_e.

For every $e \in M$, define a function $\widetilde{f}_e$ which on input x outputs the smallest $z \in M$ such that either $\langle e,x,0,z\rangle$ or $\langle e,x,1,z\rangle$ is in A. It is easy to see that $\{\widetilde{f}_e \colon e \in E\}$ is a weakly represented family of functions. Then, by DOM, there is a function $g \in S$ dominating all functions $\widetilde{f}_e$, $e \in E$. Observe that then $\{C_e \colon e \in M\}$, as defined by $C_e = \{x \colon \exists z \leq g(x)\,[\langle e,x,1,z\rangle \in A]\}$, is a uniform family of sets.

Let $e \in M$. If $e \in E$ then $C_e \subseteq B_e$ by definition, and, as g dominates all $\widetilde{f}_e$ with $e \in E$, there is a b_1 such that all $x > b_1$ satisfy $C_e(x) = B_e(x)$. If, on the other hand, $e \notin E$ then there is a b_0 such that $\langle e, x, 1, z\rangle \notin A$ for all $x > b_0$ and all $z \in M$; thus C_e is finite. Let $b = b_0$ if $e \notin E$, and let $b = b_1$ otherwise.

Now, by COH, there is an infinite set $D \in S$ such that, for every $e \in M$, there is a bound b' satisfying that for all $x, x' > b'$, if $x, x' \in D$ then $C_e(x) = C_e(x')$.

Thus for all $x, x' > \max(b, b')$, if $x, x' \in D$ then $B_e(x) = B_e(x')$. That is, D witnesses that the requirements of COHW concerning $\mathcal{F} = \{B_e : e \in E\}$ are satisfied. As $\mathcal{F}$ was arbitrary, COHW is satisfied in general. □

Note that a similar result also holds for WKL_0 in place of DOM, that is, over $\mathsf{RCA}_0 + \mathsf{WKL}_0$, COH implies COHW. The reason is that WKL_0 proves that every weakly represented family $\mathcal{F}$ of sets is contained in a uniformly represented family $\mathscr{G}$ of sets, from which it follows that COH is equivalent to COHW.

Hirschfeldt [16, Open Question 9.18] asked if $\mathsf{RCA}_0 + \mathsf{CADS}$ implies COH. Here CADS is the principle that whenever $\sqsubset \in S$ is a linear ordering on M then there is an infinite set $A \in S$ such that for every $i \in A$ there is a $k \in M$ such that either all $j \in A$ satisfy $k < j \to i \sqsubseteq j$ or all $j \in A$ satisfy $k < j \to j \sqsubset i$. We now show that $\mathsf{RCA}_0 + \mathsf{DOM} \vdash \mathsf{CADS}$, and thus an affirmative answer to Hirschfeldt's question would also prove $\mathsf{RCA}_0 + \mathsf{DOM} \vdash \mathsf{COH}$. Note that $\mathsf{RCA}_0 + \mathsf{DOM}$ does not imply the closely related principle SADS (Hirschfeldt [16, Definition 9.16]); this is because $\mathsf{SADS} \vdash \mathsf{B}\Sigma_2$ while $\mathsf{DOM} \nvdash \mathsf{B}\Sigma_2$.

Theorem 47. *Over* RCA_0, DOM *implies* CADS.

Proof. Let a linear ordering $\sqsubset \in S$ be given and define for each $e \in M$ the function f_e via $f_e(i) = \min\{j \geq i : e \sqsubseteq j\}$. Note that f_e is total iff there are infinitely many j with $e \sqsubseteq j$. Then $\mathcal{F} = \{f_e : f_e \text{ total}\}$ forms a weakly represented family and so all functions $f \in \mathcal{F}$ are dominated by a single function $g \in S$. Let

$$h(i) = \max_{\sqsubseteq}\{j : i \leq j \leq i + g(i)\}$$

and let A be the range of h. Then $i \in A \Leftrightarrow i \in \{h(0), h(1), \ldots, h(i)\}$.

Now let e be given. If there are infinitely many j with $e \leq j$ then, for almost all i, there is a $j \in \{i, i+1, \ldots, i + g(i)\}$ with $e \sqsubseteq j$; it follows that $e \sqsubseteq h(i)$. If there are only finitely many such j then $h(i) \sqsubset e$ for almost all i. Thus for each e it holds that either almost all $j \in A$ satisfy $e \sqsubseteq j$ or almost all $j \in A$ satisfy $j \sqsubset e$.

As the choice of $\sqsubset$ was arbitrary, CADS holds. □

8 DOM does not imply SRT^2_2

We now construct an ω-model witnessing that DOM does not imply SRT^2_2. We require the following lemma.

Lemma 48. *Let A be Martin-Löf random relative to Ω. Then A does not compute any infinite subset of Ω or $\overline{\Omega}$.*

Proof. Without loss of generality, assume that A computes an infinite subset G of Ω; the case $\overline{\Omega}$ is symmetric. By Theorem 40 we have that Ω is Martin-Löf random relative to A. Since $G \leq_{\mathrm{T}} A$, Ω is also Martin-Löf random relative to G. Let $(b_i : i \in \omega)$ be a strictly monotone listing of the elements of G. Then it is easy to see that the sequence $(U_n : n \in \omega)$ defined via

$$U_n = [\{\sigma \in \{0,1\}^{b_n+1} : \sigma(b_i) = 1 \text{ for all } 0 \leq i \leq n\}]$$

is a G-Martin-Löf test covering Ω, contradiction. □

Theorem 49. DOM *does not imply* SRT^2_2.

Proof. We construct an ω-model of $\mathsf{DOM} + \neg\mathsf{SRT}^2_2$. To achieve this, we will use a result of Chong, Lempp and Yang [6] and Cholak, Jockusch and Slaman [5] who proved that SRT^2_2 is equivalent to the following principle D^2_2:

> For every Δ^0_2 set $G \subseteq \omega$ there exists an infinite $A \subseteq \omega$ such that $A \subseteq G$ or $A \subseteq \overline{G}$.

To ensure $\neg\mathsf{SRT}^2_2$ it is therefore enough to ensure $\neg\mathsf{D}^2_2$. To this end, for all $n \in \omega$, let $A_n = \Omega^{\emptyset'} \oplus \Omega^{\emptyset''} \oplus \ldots \oplus \Omega^{\emptyset^{(n+1)}}$, where $\emptyset^{(i)}$ is the i-th Turing jump for $i \in \omega$. Now let

$$S = \{A \subseteq \omega : A \leq_{\mathrm{T}} A_n \text{ for some } n \in \omega\}$$

and let $\mathcal{M} = (\omega, S, +, \cdot, 0, 1)$. As $\emptyset' =_{\mathrm{T}} \Omega$ we have that $\Omega^{\emptyset'}$ is Martin-Löf random relative to Ω, and by repeated application of Theorem 40 it follows that, for any $n \in \omega$, A_n is Martin-Löf random relative to Ω. By Lemma 48 we obtain that no set in S computes an infinite subset of Ω or $\overline{\Omega}$. But since Ω is Δ^0_2 this implies $\neg\mathsf{D}^2_2$.

To see that DOM is satisfied by $\mathcal{M}$ let a weakly represented family $\mathcal{F}$ represented by $A \leq_{\mathrm{T}} A_i$ for $i \in \omega$ large enough be given. Note that A_{i+1} is high relative to A_i, and that it therefore computes a function g dominating all functions computable from A, in particular g dominates all $f \in \mathcal{F}$. As $\mathcal{F}$ was arbitrary, this establishes DOM. □

9 Restricted Π^1_2-conservativeness of DOM over RCA_0

In this section we will prove that given any restricted Π^1_2-sentence φ,

> if $\mathsf{DOM} + \mathsf{RCA}_0 \vdash \varphi$, then $\mathsf{RCA}_0 \vdash \varphi$.

Here a formula φ is called a restricted Π^1_2-sentence iff it is of the form

$$\forall X\, [\alpha(X) \rightarrow \exists Y\, [\beta(X,Y)]]$$

where X, Y are quantified variables ranging over the second order part of the model in question, α is any arithmetical formula and β is a Σ^0_3-formula. We begin by introducing the following concepts.

Definition 50. Given a structure $\mathcal{M} = (M, S, +, \times, 0, 1, <)$ of second order arithmetic and $g \subseteq M$, let $\mathcal{M}_g$ be the $\mathcal{L}_2$-structure $(M, S \cup \{g\}, +, \times, 0, 1, <)$ and $\mathcal{M}[g]$ be the $\mathcal{L}_2$-structure $(M, \Delta^0_1(\mathcal{M}_g), +, \times, 0, 1, <)$ where

$$\Delta^0_1(\mathcal{M}_g) = \{X \subseteq M : X \text{ is } \Delta^0_1 \text{ definable over } \mathcal{M}_g\}.$$

Remark 51. By a result of Simpson [39, Lemma IX.1.8], for every $\mathcal{L}_2$-structure $\mathcal{M}$ and $g \subseteq M$, if $\mathcal{M}_g$ satisfies the basic axioms and $\mathsf{I}\Sigma_1$, then $\mathcal{M}[g]$ is a model of RCA_0.

Hirschfeldt [16, Proposition 7.16] proved that a statement of the form

$$\forall X\, [\vartheta(X) \to \exists Y\, [\psi(X, Y)]], \qquad (\dagger)$$

where ϑ and ψ are arithmetic formulas, is restricted Π^1_2-conservative over RCA_0 iff one can for every countable structure $\mathcal{M} = (M, S, +, \times, 0, 1, <)$ and every $X \in S$ with $\mathcal{M} \models \vartheta(X)$ find an extension $\mathcal{N} = (M, \Delta^0_1(\mathcal{M}_g), +, \times, 0, 1, <)$ such that

- $\mathcal{N} \models \mathsf{I}\Sigma_1$,
- $\mathcal{N} \models \psi(X, g)$,
- for every Σ_3-formula $\rho(Y, Z)$ whose free variables are exactly $\{Y, Z\}$ and every $Z \in S$, if $\mathcal{N} \models \exists Y\, [\rho(Z, Y)]$ then $\mathcal{M} \models \exists Y\, [\rho(Z, Y)]$.

Note that DOM is given by a Π^1_2-formula of the form $(\dagger)$ where $\vartheta(X)$ states that X represents a weakly represented family $\mathcal{F}$ of functions and $\psi(X, Y)$ states that Y dominates every function in $\mathcal{F}$. Thus we can use Hirschfeldt's criterion to prove the following theorem.

Theorem 52. DOM *is restricted* Π^1_2*-conservative over* RCA_0*.*

Proof. We use the coinfinite extension method of Kleene and Post [26], Lacombe [28] and Spector [40] as described by Odifreddi [35, Theorem V.4.3] to prove the result; these methods will be adjusted to work on countable models of arithmetic. The function g above will be constructed by an induction over the natural numbers for which we use a list covering the countable set of requirements listed below; furthermore, we use that there is a cofinal ascending sequence $a_0, a_1, \ldots$ of elements of M and that every ascending sequence $b_0, b_1, \ldots$ with $b_n \geq a_n$ for all n is also cofinal. The following invariant will be maintained at all stages n:

At the beginning of stage n, the function g is defined for all $\langle x, y \rangle$ with $x < b_n$ and its extension $\tilde{g}$ is in S where $\tilde{g}$ takes the value 0 on those places where g is not yet defined. Furthermore, when b_{n+1} is chosen to satisfy the requirement, it is done in such a way that $\max\{b_n, a_{n+1}\} \leq b_{n+1}$.

The ideas of this construction combine the original result of Spector as presented by Odifreddi with ideas of Hirschfeldt [16, Chapters 6 and 7]. The requirements used are the three items below; they are stated together with a description of how they are realised at the stage n where they get attention; and as there are only countably many of these (all parameters range over S and M), there is a non-effective enumeration of these conditions by natural numbers.

- For all $X \in S$ and Turing reductions $F \in S$ and $u \in M$, if $F^{g \oplus X}$ is total and $\{0,1\}$-valued and for all v there is an w with $F^{g \oplus X}(u,v,w) = 1$ then there is an $h \in S$ with $F^{h \oplus X}$ being total and $\{0,1\}$-valued and for all v there is a w with $F^{h \oplus X}(u,v,w) = 1$.

  This requirement is satisfied as follows: Let $c_0 = 0$ and η_0 be the everywhere undefined function; for $m = 0,1,\ldots$ we search for a finite function η_{m+1} and a value c_{m+1} such that the following conditions hold:
  - $c_{m+1} > m$;
  - the domain of η_{m+1} is $\{\langle x,y\rangle : x,y < c_{m+1}\}$ and η_{m+1} can be coded using an element of M;
  - all $x,y < c_m$ satisfy $\eta_{m+1}(\langle x,y\rangle) = \eta_m(\langle x,m\rangle)$;
  - all $x < \min\{b_n, c_{m+1}\}$ and all $y < c_{m+1}$ satisfy $\eta_{m+1}(\langle x,y\rangle) = g(\langle x,y\rangle)$;
  - $F^{\eta_{m+1} \oplus X}(u,m,w) = 1$ for some $w < c_{m+1}$ without F querying the first component of the join $\eta_{m+1} \oplus X$ outside the domain of η_{m+1};
  - $F^{\eta_{m+1} \oplus X}(\tilde{u},\tilde{v},\tilde{w})$ terminates with a value from $\{0,1\}$ without querying outside the domain of η_{m+1} for all $\tilde{u},\tilde{v},\tilde{w} < c_m$.

  By $\mathsf{I}\Sigma_1$ there are only two cases.

  Case 1. The construction goes through for all m yealding in the limit a total extension h of the part of g constructed so far such that $F^{h \oplus X}$ is total and $\{0,1\}$-valued and the value u satisfies that for every v there is a w with $F^{h \oplus X}(u,v,w) = 1$. As the part of g constructed prior to stage n is the restriction of a function in S to a domain in S, the so constructed h is also in S. In this case the requirement is satisfied and one selects $b_{n+1} = \max\{a_{n+1}, b_n\}$ and defines for all x with $b_n \leq x < b_{n+1}$ and all y that $g(\langle x,y\rangle) = 0$.

  Case 2. The construction progresses until it reaches an m for which the extension η_{m+1} cannot be found; the existence of such an m in the case that not all m are used follows from $\mathsf{I}\Sigma_1$. Now any common extension $\tilde{g}$ of the part of g built so far and of η_m found so far satisfies that either $F^{\tilde{g} \oplus X}$ is undefined or above 2 for some inputs $(\tilde{u},\tilde{v},\tilde{w})$ or $v = m$ satisfies that there is no w with $F^{\tilde{g} \oplus X}(u,v,w) = 1$. Now one extends g as follows: $b_{n+1} = \max\{a_{n+1}, b_n, c_m\}$ and for all $x < b_{n+1}$ and all $y \in M$ one defines if $x < b_n$ then $g(\langle x,y\rangle)$ is defined as done previously else if $x < c_m$ and $y < c_m$ then $g(\langle x,y\rangle) = \eta_m(\langle x,y\rangle)$ else $g(\langle x,y\rangle) = 0$.

  Note that in both cases, one extends the function with finite case distinction between finite functions codable in S and existing functions which are restrictions of functions in S to a domain in S; then the newly extended part of g is also a restriction of a function in S to the domain $\{\langle x,y\rangle : x < b_{n+1}\}$, which is a set in S.
- For all $X \in S$ and Turing reductions $F \in S$, the range of $F^{g \oplus X}(M)$ has a minimum.

  This requirement is satisfied as follows: Let $c_0 = 0$ and η_0 be the everywhere undefined function; for $m = 0,1,\ldots$ we search for a finite function η_{m+1} and a value c_{m+1} such that the following conditions hold:
  - $c_{m+1} > m$;

  - the domain of η_{m+1} is $\{\langle x,y\rangle : x,y < c_{m+1}\}$ and η_{m+1} can be coded using an element of M;
  - all $x,y < c_m$ satisfy $\eta_{m+1}(\langle x,y\rangle) = \eta_m(\langle x,m\rangle)$;
  - all $x < \min\{b_n, c_{m+1}\}$ and all $y < c_{m+1}$ satisfy $\eta_{m+1}(\langle x,y\rangle) = g(\langle x,y\rangle)$;
  - if $m = 0$ then $v = F^{\eta_{m+1}\oplus X}(w)$ is defined for some w; else there is a w such that $F^{\eta_{m+1}\oplus X}(w)$ is defined and bounded by $v-m$ with the v from the case $m=0$; furthermore, the computation of $F^{\eta_{m+1}\oplus X}(w)$ does not query any elements of the η_{m+1}-part of the join $\eta_{m+1}\oplus X$ except those where η_{m+1} is defined.

  By $\mathsf{I}\Sigma_1$ this construction runs only up to some m; this m is at most v for the v chosen at $m=0$. The reason for this is that afterwards the requirement would be that there is a w for which $F^{\eta_{m+1}\oplus X}(w)$ is defined and negative; however, this is not allowed as the outputs of the function are all in M. So now let m be the maximum number for which η_m is defined, this number exists by $\mathsf{I}\Sigma_1$. Then any total common extension $\tilde{g}$ of the part of g constructed so far and of η_m satisfies that the so defined function does not take values below $v-m$ while the value $v-m$ exists by the existence of η_m.

  Let $b_{n+1} = \max\{a_{n+1}, b_n, c_m\}$ and define that $g(\langle x,y\rangle)$ with $b_n \le x < b_{n+1}$ takes the value $\eta_m(\langle x,y\rangle)$ in the case that $x,y < c_m$ and takes the value 0 otherwise. The so chosen extension is again the restriction of a function in S to the domain $\{\langle x,y\rangle : x < b_{n+1}\}$ which is also a set in S. Furthermore, the function computed by F from g takes a minimum and so this necessary requirement towards satisfying $\mathsf{I}\Sigma_1$ in the model $\mathcal{M}\cup\{g\}$ is satisfied.
- For all $f \in S$ there is a $x \in M$ such that $\forall y\,[g(\langle x,y\rangle) = f(y)]$.

  This is the easiest requirement to satisfy: If $f \in S$ is the function in question, then we select $b_{n+1} = \max\{a_{n+1}, b_n+1\}$ and define $g(\langle x,y\rangle) = f(y)$ for all $y \in M$ and all x with $b_n \le x < b_{n+1}$.

The last requirement ensures that g codes a uniform family which contains all functions contained in any weakly represented family in $\mathcal{M}$. Thus it follows that the function $h(y) = 1 + (\sum_{x=0}^{y} g(\langle x,y\rangle))$ dominates every weakly represented family in the model $\mathcal{M}$ and h is clearly a function in $\mathcal{M}\cup\{g\}$. It follows that the preconditions of Proposition 7.16 by Hirschfeldt [16] are satisfied and therefore DOM is restricted Π^1_2-conservative over RCA_0. □

Theorem 53. DOM *is not* Π^1_1*-conservative over* $\mathsf{RCA}_0 + \mathsf{B}\Sigma_2$*.*

Proof. Hölzl, Jain and Stephan [19, Theorem 20] showed that over $\mathsf{RCA}_0 + \mathsf{B}\Sigma_2$, DOM implies $\mathsf{I}\Sigma_2$. So we have that $\mathsf{DOM} + \mathsf{RCA}_0 + \mathsf{B}\Sigma_2 \vdash \mathsf{I}\Sigma_2$, while it is well-known that $\mathsf{RCA}_0 + \mathsf{B}\Sigma_2 \nvdash \mathsf{I}\Sigma_2$. But since $\mathsf{I}\Sigma_2$ can be formalised by a Π^1_1-statement, DOM is not Π^1_1-conservative over $\mathsf{RCA}_0 + \mathsf{B}\Sigma_2$. □

This result stands in contrast to the result of Chong, Slaman and Yang [7] that COH is Π^1_1-conservative over $\mathsf{RCA}_0 + \mathsf{B}\Sigma_2$. Furthermore, as DOM implies AVOID, MEET and BI, we obtain the following immediate consequence.

Corollary 54. *The following are restricted* Π^1_2*-conservative over* RCA_0*.*

(a) AVOID
(b) MEET
(c) BI

Finally, the results in this section provide another proof of Theorem 49; the argument being that $\mathsf{RCA}_0 + \mathsf{DOM} + \neg\mathsf{B}\Sigma_2$ has a model; and that such a model cannot satisfy SRT^2_2, as this would contradict the result of Cholak, Jockusch and Slaman [5] that $\mathsf{RCA}_0 + \mathsf{SRT}^2_2 \vdash \mathsf{B}\Sigma_2$.

10 Almost Disjointness and Independence

In this section we prove that in ω-models MAD and MIND coincide with the negations of previously known principles.

Theorem 55. *An ω-model satisfies* MAD *iff it does not satisfy* DOM*.*

Proof. ($\Rightarrow$): Let $\mathcal{F}$ be a weakly represented family of sets represented by $A \in S$ that is almost disjoint. Suppose that DOM holds; we will show that this implies $\neg$MAD.

Assume without loss of generality that for the characteristic function f of every set in $\mathcal{F}$ there is a unique $e \in \omega$ such that f is weakly represented by A_e (where A_e is as in Definition 5). Indeed, this can be achieved by replacing A with a set A' derived from it, where A' and $A'_e = \{n \colon \langle e, n\rangle \in A'\}$ are such that whenever f'_e (the function weakly represented by A'_e) looks identical to f'_d for some $d < e$, the enumeration of elements into A'_e is suspended; this way, should there indeed be a $d < e$ with $f_e = f_d$ in the limit, then f'_e will become non-total, and A'_e will not weakly represent any function in $\mathcal{F}$.

As a consequence of the previous assumption, if, for some $d \neq e$, A_d and A_e weakly represent the characteristic functions of sets $F \in \mathcal{F}$ and $G \in \mathcal{F}$ respectively, then $F \cap G$ is finite.

Let $(\varphi^A_e \colon e \in \omega)$ be an enumeration of all A-recursive functions.

Claim. There is a function $g \in S$ that dominates every A-recursive function in the following sense: For every total φ^A_e and almost all n it holds that

$$g(n+1) > \varphi^A_e(g(n)).$$

Proof. Consider the weakly represented family of all A-recursive functions, and apply DOM to obtain a function $\widehat{g}$ dominating it. Without loss of generality, assume that $\widehat{g}$ is strictly increasing, let $g(0) = 0$ and define for all n that $g(n+1) = \widehat{g}(g(n))$. $\diamond$

Let $h(x, n)$ be the least number d such that either $\varphi^A_d(x)[g(n+2)]\downarrow = 1$ or $d = n$. Let B be the set consisting of numbers $b_n \in \omega$, $n \in \omega$, with $g(n) < b_n < g(n+1)$ and $h(b_n, n) \geq h(x, n)$ for all x with $g(n) < x < g(n+1)$.

Informally, for an element x, the value $h(x)$ tells us that x does not seem to show up in those sets that have characteristic functions who have an A-recursive index up to $h(x)$; of course this can only be determined given an enumeration timebound, which is provided by the dominating function g here. Then B picks elements where this number is as large as possible.

More formally, note that by the choice of g, if φ_d^A coincides with the characteristic function f_e of a set in $\mathcal{F}$, then for almost all n there is an x with $g(n) < x < g(n+1)$ such that $f_e(x)[g(n+2)]\downarrow = 1$ and, due to the almost disjointness, for all $d < e$ either $f_d(x)\uparrow$ or $f_d(x)\downarrow = 0$. As by construction B consists only of numbers of this type, for almost all n it holds that $b_n \notin A_d$ for $d < e$ and therefore the set B has finite intersection with every $C \in \mathcal{F}$. Thus MAD is not satisfied.

($\Leftarrow$): Let $\mathcal{M} = (\omega, S, +, \cdot, 0, 1)$ be an ω-model of $\neg\mathsf{DOM}$. We assume without loss of generality that S contains no high set; otherwise carry out the construction below relative to an oracle relative to which no high set in S exists.

The fact that we don't know which indeces e describe total recursive functions φ_e is a complication in the construction that follows. To circumvent this issue, we take advantage of the possibilities that the concept of weakly represented families offer, namely that partial information about functions is ignored when defining such a family; only total functions are considered a member of the family. Using this, we build a recursive numbering of partial-recursive functions such that the total functions appearing in it are all $\{0, 1\}$-valued and when interpreted as characteristic functions of sets, the collection of these sets is a maximal almost disjoint family.

Let $(\varphi_e\colon e \in \omega)$ be an enumeration of all recursive functions. First we build the uniformly recursive helper procedures $\psi_{c_0,c_1,\dots,c_e}$ for all $e \in \omega$ with $c_d \in \{0, 1, \dots, \infty\}$ for $d \leq e$. We call $(c_0, c_1, \dots, c_e)$ *true parameters* if, for all $d \leq e$, c_d is the minimal i such that $\varphi_d(i)\uparrow$ if such an i exists, and $c_d = \infty$ if φ_d is total.

The procedure $\psi_{c_0,c_1,\dots,c_e}$ has three states: *wait*, *success*, and *aborted*. When we define the enumeration of the characteristic function of A_e below, we will only enumerate a new function value whenever $\psi_{c_0,c_1,\dots,c_e}$ is in state *success*. The idea is that this will only happen infinitely often, when $(c_0, c_1, \dots, c_e)$ are the true parameters. If $(c_0, c_1, \dots, c_e)$ are not true parameters, then $\psi_{c_0,c_1,\dots,c_e}$ will either be stuck in state *wait* forever, or it will enter state *aborted* and stay in it forever. Then the true parameters will be the only parameters used to define A_e.

To achieve what we just described, we proceed as follows: $\psi_{c_0,\dots,c_e}$ starts in state *wait* and runs the following $e + 1$ parallel procedures:

- For all $d \leq e$, the computations $\varphi_d(c_d)$, $d \leq e$, are run in parallel. If one of them ever terminates, then by definition $(c_0, c_1, \dots, c_e)$ are not true parameters. Then $\psi_{c_0,\dots,c_e}$ stops all computations, enters state *aborted*, and remains in this state permanently.
- In a single procedure, all computations $\varphi_d(c)$ with $d \leq e$ and $c < c_d$ are run *sequentially* and in order ascending with $\langle d, c\rangle$. While one of the computations runs, $\psi_{c_0,\dots,c_e}$ is in state *wait*. Every time one of the computations $\varphi_d(c)$ terminates, $\psi_{c_0,\dots,c_e}$ enters state *success*. If (d, c) was the last pair of parameters as above (which can only happen if all c_d, $d \leq e$, are finite) then remain

in state *success* permanently. Otherwise enter state *wait* again, and continue with the next pair (d', c'), that is, with the smallest pair as above such that $\langle d', c'\rangle > \langle d, c\rangle$.

Note that this arrangement ensures that $\psi_{c_0,\ldots,c_e}$ is in state *success* at infinitely many stages if and only if $(c_0, c_1, \ldots, c_e)$ are the true parameters.

We can now describe how to produce a maximal almost disjoint family. In parallel, for all $e \in \omega$ and all possible sets of parameters $(c_0, c_1, \ldots, c_e)$ we run the following procedure.[5]

Run $\psi_{c_0,\ldots,c_e}$ step by step.
At every stage, check if $\psi_{c_0,\ldots,c_e}$ is currently in state *success*.
If so, let m be the smallest number not in $A_0 \cup A_1 \cup \ldots \cup A_{e-1}$, and check whether

$$m = n + \varphi_e(0) + \varphi_e(1) + \ldots + \varphi_e(n) \text{ for some } n. \qquad (*)$$

If not, enumerate m into A_e.

Note that if $(c_0, c_1, \ldots, c_e)$ are the true parameters, then checking $(*)$ is recursive, and the procedure never gets stuck. This finishes the construction.

We need to prove that the weakly represented family $\{A_n : n \in \omega\}$ constructed by this procedure is maximal almost disjoint. First note that for each $e \in \omega$ the complement of $A_0 \cup \ldots \cup A_e$ is infinite and contains at most n elements below $\varphi_e(n)$. Furthermore, A_e is disjoint with all A_d for $d < e$. As a consequence, $\{A_n : n \in \omega\}$ is almost disjoint.

It remains to show that $\{A_n : n \in \omega\}$ is also maximal almost disjoint. To see this let B be an infinite non-high set. Then there is a recursive function φ_e such that, for infinitely many n, there are more than $2n$ elements of B below $\varphi_e(n)$. It follows that the intersection of B with $A_0 \cup \ldots \cup A_e$ is infinite and therefore $B \cap A_d$ is infinite for some $d \leq e$. This completes the proof. □

Theorem 56. *An ω-model satisfies* MIND *iff it does not satisfy* BI.

Proof. $(\Rightarrow)$: As before, for a set $C \subseteq \omega$, let us write C^0 for C and C^1 for $\omega \setminus C$.

Let a weakly represented family $\mathcal{F}$ of sets represented by A be given. Also fix any collection $\{A_0, \ldots, A_{n-1}\} \subseteq \mathcal{F}$ and any string $\sigma \in 0,1^n$, as well as a set B which is biimmune relative to A. Observe that the set $\widehat{A} = \bigcap_{i<n} A_i^{\sigma(i)}$ is A-recursive. Then B's biimmunity relative to A implies that both $\widehat{A} \cap B$ and $\widehat{A} \cap \overline{B}$ are infinite.

As $\{A_0, \ldots, A_{n-1}\}$ was arbitrary, it follows that $\mathcal{F} \cup \{B\}$ is still an independent family, which contradicts the assumption that $\mathcal{F}$ is maximal independent.

[5] Note that to simplify notation, we do not explicitly define total characteristic functions of the sets A_e, $e \in \omega$, or the enumeration of a set that represents these functions as a weakly represented family. But since the elements of every A_e are enumerated in increasing order by the given procedure, it is easy to convert it into one defining the enumeration of such a set.

($\Leftarrow$): Similarly to the proof of the previous theorem, we work with lists of parameters where true parameters define sets in the maximal independent family that we need to construct. So assume that an ω-model $\mathcal{M} = (\omega, S, +, \cdot, 0, 1)$ and a set $A \in S$ are given such that no set $B \in S$ is biimmune relative to A; to simplify notation, we assume that A is recursive; otherwise carry out the construction relative to A.

In the following, a stream is an infinite sequence of natural numbers in strictly ascending order. Each stream will be indexed with a string; the range of streams x^σ, x^τ is disjoint if σ, τ are incomparable as strings, and the range of x^σ is a superset of the range of x^τ when σ is a prefix of τ.

We begin the construction with the initial stream x^ε which is the sequence of all natural numbers, that is, $x^\varepsilon_n = n$ for all $n \in \omega$. We now describe how to define the streams x^σ, for strings σ, and then we argue that the set $\{E_e \colon e \in \omega\}$ with

$$E_e = \{m \colon \exists \sigma \in \{0,1\}^e \; \exists n \; [m = x^{\sigma 1}_n]\}$$

is a weakly represented family that is maximal independent. To define the streams x^σ, for strings σ, proceed as follows for all $e \in \omega$:

Let $R_e(n)$ be defined and let it equal $\varphi_e(n)$ iff the values $\varphi_e(0), \ldots, \varphi_e(n)$ are defined and in $\{0,1\}$; let $R_e(n)$ be undefined if there is $m \leq n$ where $\varphi_e(m)$ is undefined or defined and at least 2. We define a function $\eta_e \colon \{0,1\}^e \to \{0,1\}$ as follows:

(a) If R_e is total and there exist both infinitely many $n \in \text{range}(x^\sigma)$ such that $R_e(n) = 0$ and infinitely many $n \in \text{range}(x^\sigma)$ such that $R_e(n) = 1$, then let, for $a = 0, 1$ and all $n \in \omega$, $x^{\sigma a}_n$ be the n-th element m of x^σ with $R_e(m) = a$. Informally this means that x^σ is split into $x^{\sigma 0}$ and $x^{\sigma 1}$ according to the values of R_e. Furthermore, let $\eta_e(\sigma) = 1$.

(b) Else let $x^{\sigma 0}_n = x^\sigma_{2n}$ and $x^{\sigma 1}_n = x^\sigma_{2n+1}$ for all $n \in \omega$. Informally this means that x^σ is split evenly into $x^{\sigma 0}$ and $x^{\sigma 1}$. Furthermore, let $\eta_e(\sigma) = 0$.

Note that η_e stores the information for which σ of length e cases (a) and (b) applied, respectively. This finishes the construction.

As an auxiliary notion need for the verification we define

> t_d as the maximum n such that $n = 0$ or one can find a $\tau \in \{0,1\}^d$ and x^τ_i, x^τ_j with $\eta_d(\tau) = 0$, $n = \min\{x^\tau_i, x^\tau_j\}$, $R_d(x^\tau_i) = 0$ and $R_d(x^\tau_j) = 1$.

Note that for a given $\tau \in \{0,1\}^d$ the statement $\eta_d(\tau) = 0$ means that case (b) applied to τ above, and that $R_d(k)$ is the same value for all $k > t_d$ with $k \in \text{range}(x^\tau)$ such that $R_d(k)$ is defined. The same holds for all other τ of length d with $\eta_d(\tau) = 0$.

The verification consists of establishing the following two claims.

Claim. $\{E_e \colon e \in \omega\}$ is maximal independent.

Proof. Note that for $\sigma \in \{0,1\}^e$, range(x^σ) is the intersection of all E_d with $d < e$ and $\sigma(d) = 1$ and all $\overline{E_d}$ with $d < e$ and $\sigma(d) = 0$. Further note that by construction range(x^σ) has infinite cardinality. Thus $\{E_e : e \in \omega\}$ is independent.

To see that it is also maximal independent, consider any set B. As B cannot be biimmune it either has an infinite recursive subset or $\overline{B}$ has an infinite recursive subset; let e be such that R_e is the characteristic function of this set which we also denote R_e, slightly abusing notation. Now, for some $\sigma \in \{0,1\}^e$, x^σ has infinite intersection with R_e and therefore by construction almost all elements in range$(x^{\sigma 1})$ are also elements of R_e.

So for the Boolean combination of $E_0, E_1, \ldots, E_e$ that equals range$(x^{\sigma 1})$ we have that it either equals an infinite subset of B or an infinite set disjoint with B. Thus $\{E_e : e \in \omega\} \cup \{B\}$ cannot be independent. ◇

Claim. $\{E_e : e \in \omega\}$ is a weakly represented family.

Proof. Recall that the parameters η_e, $e \in \omega$, store for which σ of length e which of the two cases (a) and (b) was applied during the construction. The following construction is described for arbitrary parameter sets $(\widetilde{\eta}_0, \widetilde{\eta}_1, \ldots, \widetilde{\eta}_e, s)$. As in the proof of Theorem 55, for each e, the construction below will only define a set E_e if $(\widetilde{\eta}_0, \widetilde{\eta}_1, \ldots, \widetilde{\eta}_e) = (\eta_0, \eta_1, \ldots, \eta_e)$ and if s is a sufficiently large timebound. For all other parameter sets, the construction will get stuck eventually.

More formally, let $c = (\widetilde{\eta}_0, \widetilde{\eta}_1, \ldots, \widetilde{\eta}_e, s)$ be given, where $\widetilde{\eta}_d \in \{0,1\}^{2^d}$ for all $d \leq e$ and let $s \in \omega$. We describe how to inductively construct streams $\widetilde{x}^\sigma$ for each string σ based on c:

(a) If $\widetilde{\eta}_e(\sigma) = 1$, then let, for $a = 0, 1$ and all $n \in \omega$, $\widetilde{x}^{\sigma a}_n$ be the n-th element m of $\widetilde{x}^\sigma$ with $R_e(m) = a$.
(b) Else let $\widetilde{x}^{\sigma 0}_n = \widetilde{x}^\sigma_{2n}$ and $\widetilde{x}^{\sigma 1}_n = \widetilde{x}^\sigma_{2n+1}$ for all $n \in \omega$.

In other words, we try to mimic the previous construction, hoping that c is a set of true parameters. Now let

> $\widetilde{t}_d$ be the maximum $n \leq s$ such that either $n = 0$ or such that one can find within time s some $\tau \in \{0,1\}^d$ and some $\widetilde{x}^\tau_i$, $\widetilde{x}^\tau_j$ with $\widetilde{\eta}_d(\tau) = 0$ and $n = \min\{\widetilde{x}^\tau_i, \widetilde{x}^\tau_j\}$ and such that $R_d(\widetilde{x}^\tau_i)$, $R_d(\widetilde{x}^\tau_j)$ become defined within time s and $R_d(\widetilde{x}^\tau_i) \neq R_d(\widetilde{x}^\tau_j)$.

We now need to define an algorithm that uniformly from c produces a partial function F_c such that on the one hand, if for all $d \leq e$, $\widetilde{\eta}_d = \eta_d$, then for sufficiently large s, $\widetilde{t}_d = t_d$ for all $d \leq e$ and F_c is the characteristic function of E_e; and such that on the other hand, if for some $d \leq e$, $\widetilde{\eta}_d \neq \eta_d$ or $\widetilde{t}_d \neq t_d$, then F_c is only defined on finitely many inputs. The algorithm to compute $F_c(n)$ is as follows:

(1) Compute $\widetilde{x}^\sigma_0, \ldots, \widetilde{x}^\sigma_n$ for all $|\sigma| \leq e + 1$.
(2) Determine the unique $\sigma \in \{0,1\}^{e+1}$ such that there is an m with $\widetilde{x}^\sigma_m = n$.
(3) Search for n computation steps for a d and $\tau \in \{0,1\}^d$ such that $\widetilde{\eta}_d(\tau) = 0$ and for some $\widetilde{x}^\tau_i, \widetilde{x}^\tau_j > \widetilde{t}_d$ with $R_d(\widetilde{x}^\tau_i) \downarrow = 0$ and $R_d(\widetilde{x}^\tau_j) \downarrow > 0$. (If we find these, then $\widetilde{\eta}_d$ must be wrong, or $\widetilde{t}_d < t_d$.)

(4) If the computations in (1) terminate, the search in (2) is successful, and the search in (3) is *un*successful, then output $F_c(n) = \sigma(e)$, else let $F_c(n)$ be undefined.

We verify that this algorithm behaves as required. First assume that all $\widetilde{\eta}_d = \eta_d$. Then all $\widetilde{x}^\sigma$ with $|\sigma| \leq e+1$ are equal to x^σ. Furthermore, for large enough s the definition of $\widetilde{t}_d$ ensures that $\widetilde{t}_d = t_d$. Then the algorithm above produces a total function F_c and by (4) we have that F_c is the characteristic function of E_e.

If, on the other hand, there is a d such that either $\widetilde{\eta}_d \neq \eta_d$ or $\widetilde{t}_d < t_d$ then let d be the least such d. Note that then for σ with $|\sigma| \leq d$, it holds that $\widetilde{x}^\sigma = x^\sigma$. We argue that F_c is not total; there are several cases to consider.

- If $\eta_d(\tau) = 1$ and $\widetilde{\eta}_d(\tau) = 0$ for some $\tau \in \{0,1\}^d$, then there are infinitely many elements of x^τ for which R_e takes the value 1 and infinitely many for which R_e takes the value 0. However, as $\widetilde{\eta}_d(\tau) = 0$, for large enough n, some of these elements will be found in step (3) of the above algorithm, and $F_c(n)$ will be undefined.
- If $\eta_d(\tau) = 0$ and $\widetilde{\eta}_d(\tau) = 1$ for some $\tau \in \{0,1\}^d$, then by definition the streams $\widetilde{x}^{\tau 0}$ and $\widetilde{x}^{\tau 1}$ are defined from $\widetilde{x}^\tau = x^\tau$ by splitting according to the values of R_d; however, since $\eta_d(\tau) = 0$, one of $\widetilde{x}^{\tau 0}$ and $\widetilde{x}^{\tau 1}$ will then only contain finitely many elements. Then for sufficiently large n the algorithm will get stuck in step (1) when calculating $F_c(n)$.
- If $\widetilde{t}_d < t_d$, and the two previous cases do not apply, then $\widetilde{x}^{\tau a} = x^{\tau a}$ for all $\tau \in \{0,1\}^d$ and $a \in \{0,1\}$, and for sufficiently large n, the algorithm will in step (3) find the values $\widetilde{x}^\tau_i, \widetilde{x}^\tau_j > \widetilde{t}_d$ and $F_c(n)$ will be undefined.

Thus, the construction above only produces total functions F_c if

$$c = (\eta_0, \eta_1, \dots, \eta_e, s)$$

for a sufficiently large $s \in \omega$; and in this case F_c is the characteristic function of E_e. As the construction is uniform in c, it is easy to see that $\{E_e \colon e \in \omega\}$ is a weakly represented family. ◇

This completes the proof of Theorem 56. □

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